

NONCOMMUTATIVE COARSE GEOMETRY

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ABSTRACT. We use compactifications of C^* -algebras to introduce noncommutative coarse geometry. We transfer a noncommutative coarse structure on a C^* -algebra with an action of a locally compact Abelian group by translations to Rieffel deformations and prove that the resulting noncommutative coarse spaces are coarsely equivalent. We construct a noncommutative coarse structure from a cocompact continuously square-integrable action of a group and show that this is coarsely equivalent to the standard coarse structure on the group in question. We define noncommutative coarse maps through certain completely positive maps that induce $*$ -homomorphisms on the boundaries of the compactifications. We lift $*$ -homomorphisms between separable, nuclear boundaries to noncommutative coarse maps and prove an analogous lifting theorem for maps between the metrisable boundaries of ordinary locally compact spaces.

1. INTRODUCTION

Let (X, d) be a metric space. The topology on X associated to d encodes its small-scale features. In contrast, coarse geometry studies the large-scale geometry of the metric space, disregarding any phenomena that occur only on some finite length-scale. For instance, if a finitely generated discrete group G acts on (X, d) properly, cocompactly and by isometries, then G with any word-length metric d_G is coarsely equivalent to (X, d) , that is, coarse geometry does not distinguish the metric spaces (G, d_G) and (X, d) . Quantum mechanics allows noncommutative spaces where the coordinate functions no longer commute. Is large-scale geometry possible also for such “spaces”? Intuitively, quantum phenomena only occur on small length scales. So coarse geometry for noncommutative spaces should be possible, and the physical ones should even be coarsely equivalent to ordinary spaces. We test this intuition on Rieffel deformations, which provide several toy models used in quantum physics to model quantum spacetimes. Intuitively, Rieffel deformation only operates on some finite length scale depending on the deformation parameter. So it should not affect the coarse geometry of a space.

We are going to define noncommutative coarse spaces and noncommutative coarse maps and show that Rieffel deformation yields coarsely equivalent noncommutative coarse spaces. For instance, the Moyal plane, with a canonical noncommutative coarse structure, is coarsely equivalent to the classical Euclidean plane.

How should we extend coarse geometry to noncommutative spaces? The coarse geometric features of a metric d on a space X may be encoded through the family of all subsets of $X \times X$ on which the distance function d is uniformly bounded; these subsets are called *controlled*. Two metrics on X are *coarsely equivalent* if they have the same controlled subsets. There is, however, no obvious noncommutative analogue of controlled subsets. Connes describes Riemannian metrics in noncommutative geometry through spectral triples. A spectral triple comes with a dense subalgebra – the domain of the derivation $[D, \cdot]$ – which is complete in a suitable Lipschitz

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norm, and Rieffel [17] encodes a metric on a noncommutative space through such a densely defined Lipschitz norm on a C^* -algebra. The existing literature in this direction is mostly restricted to the unital case, however, and studies the induced topology on the state space, that is, the small-scale features of the metric. Since any bounded metric space is coarsely equivalent to the one-point space, noncommutative coarse geometry is only concerned with non-unital C^* -algebras. We shall, therefore, define coarse geometric structures on C^* -algebras differently.

We work with compactifications, which are another way to capture the coarse geometry of a metric. The *Higson compactification* of a metric space (X, d) is the spectrum of the commutative C^* -algebra of all continuous functions $f: X \rightarrow \mathbb{C}$ for which the functions

$$\Delta_R f(x) := \sup\{|f(y) - f(x)| \mid d(x, y) \leq R\}$$

vanish at infinity for all $R \geq 0$; we briefly say that such functions *oscillate slowly*. Two metrics give the same Higson compactification if and only if they give the same coarse structure; this is proved by Roe [19]. There are coarse structures that do not come from a metric and which cannot be obtained back from their associated Higson compactification. But since most coarse structures of interest come either from a metric or directly from a compactification, it seems legitimate to replace coarse structures by compactifications to achieve a noncommutative generalisation.

A *compactification* for a C^* -algebra A is a unital C^* -algebra \bar{A} with an isomorphism from A onto an essential ideal in \bar{A} . Any such \bar{A} embeds uniquely into the multiplier algebra $\mathcal{M}(A)$. Hence a noncommutative compactification is equivalent to a unital subalgebra \bar{A} of $\mathcal{M}(A)$ that contains A . A *noncommutative coarse space* is a C^* -algebra together with such a compactification.

How does Rieffel deformation affect such a noncommutative coarse space? Let Ψ be a 2-cocycle on a locally compact Abelian group Γ and let Γ act continuously on a locally compact space X . Let \bar{X} be a compactification of X such that the action of Γ on X extends continuously to \bar{X} and the induced action on the boundary $\partial X := \bar{X} \setminus X$ is trivial; then we say that Γ acts on the coarse space X *by translations*. Rieffel deformation for a given cocycle Ψ is an exact functor $A \mapsto A^\Psi$ on the category of Γ - C^* -algebras; this follows from Kasprzak's description of Rieffel deformation in [11]. It fixes C^* -algebras with a trivial action. Hence the extension

$$C_0(X) \hookrightarrow C(\bar{X}) \twoheadrightarrow C(\partial X)$$

induces another extension

$$C_0(X)^\Psi \hookrightarrow C(\bar{X})^\Psi \twoheadrightarrow C(\partial X)^\Psi$$

with $C(\partial X)^\Psi = C(\partial X)$. This provides a coarse structure on the Rieffel deformed C^* -algebra $C_0(X)^\Psi$ with the same commutative boundary $C(\partial X)$. The same argument works when we deform a noncommutative coarse space $A \triangleleft \bar{A}$, provided Γ acts trivially on the boundary $\partial A = \bar{A}/A$.

The Higson corona $\partial X = \bar{X} \setminus X$ is an important invariant of a coarse space. If the boundaries are second countable, then any continuous map between the boundaries lifts to a coarse map between the interiors (Theorem 8.1). Thus the boundary is a complete invariant up to coarse equivalence when it is second countable. It is unclear, however, whether this remains true for larger boundaries such as those coming from a proper metric on the interior. We therefore want some kind of map between $C_0(X)$ and $C_0(X)^\Psi$ that induces the identity $C(\partial X)^\Psi = C(\partial X)$ between the boundary quotient C^* -algebras. We cannot expect this to be an ordinary $*$ -homomorphism: in coarse geometry, we often need discontinuous coarse maps.

Our definition of a noncommutative coarse map is inspired by a construction by Kaschek, Neumaier and Waldmann [10] for transferring states on a C^* -algebra to

a Rieffel deformation. We define a noncommutative coarse map between two noncommutative coarse spaces (A, \bar{A}) and (B, \bar{B}) as a completely positive, contractive, strictly continuous map $\varphi: A \rightarrow B$ such that the resulting strictly continuous extension $\bar{\varphi}: \mathcal{M}(A) \rightarrow \mathcal{M}(B)$ is unital, maps \bar{A} to \bar{B} , and induces a $*$ -homomorphism $\partial A \rightarrow \partial B$. Such a map is behind the transfer of states in [10]. Two noncommutative coarse maps from $A \triangleleft \bar{A}$ to $B \triangleleft \bar{B}$ are *close* if their difference maps \bar{A} into B . In Kasprzak's approach to Rieffel deformation in [11], it is easy to build noncommutative coarse maps $A \rightarrow A^\Psi$ and back whose composites are close to the identity maps on A and A^Ψ , whenever the induced group action on the boundary ∂A is trivial. These form a coarse equivalence between A and A^Ψ . In particular, this applies to the Moyal plane and the classical plane with their canonical coarse geometries.

The construction above provides rather well-behaved quantisation maps for Rieffel deformations. For the Moyal plane, this is a map from $C_0(\mathbb{R}^2)$ to $\mathbb{K}(L^2\mathbb{R})$ that is completely positive, contractive, and strictly continuous with a unital extension $C_b(\mathbb{R}^2) \rightarrow \mathbb{B}(L^2\mathbb{R})$, and such that the induced map from $C_b(\mathbb{R}^2)/C_0(\mathbb{R}^2)$ to the Calkin algebra is a $*$ -homomorphism on the huge, non-separable C^* -algebra $C(\partial\mathbb{R}^2)$. All functions that are continuous on the usual ball compactification of \mathbb{R}^2 oscillate slowly and are therefore contained in $C(\overline{\mathbb{R}^2})$. Thus shrinking the boundary gives an extension of the Moyal plane $\mathbb{K}(L^2\mathbb{R})$ by $C(\mathbb{T})$, the continuous functions on the circle. This extension is the Toeplitz extension by the results of [6]. Thus the noncommutative coarse structure on the Moyal plane $\mathbb{K}(L^2\mathbb{R})$ given by the Toeplitz C^* -algebra extension is coarsely equivalent to the coarse structure on the classical Moyal plane $C_0(\mathbb{R}^2)$ given by Rieffel deformation of the ball compactification.

The relationship between noncommutative coarse maps and ordinary coarse maps between coarse spaces is not yet clear. Any coarse map between commutative coarse spaces is close to a noncommutative coarse map as defined above, by replacing a discontinuous map by a continuous map taking values in probability measures. We failed, however, to prove the converse, that is, it is conceivable that there are more noncommutative coarse maps than ordinary coarse maps.

As another example besides Rieffel deformations, we construct a compactification of a C^* -algebra A from a cocompact continuously square-integrable group action on A . This compactification is coarsely equivalent to the group that acts. Continuously square-integrable actions are slightly more general than the “proper” actions defined by Rieffel [16]. So this construction is a noncommutative analogue of the canonical coarse structure on a cocompact proper G -space (see Example 2.1).

For noncommutative coarse spaces with separable, nuclear boundary, we prove that the boundary is a complete invariant up to coarse equivalence. In this case, any unital $*$ -homomorphism between the boundaries lifts to a noncommutative coarse map. This is proved using quasi-central approximate units.

2. COMPACTIFICATIONS AS COARSE STRUCTURES

A coarse structure on a locally compact space X is a family of subsets of $X \times X$, called *controlled*, subject to some axioms (see [19]). For instance, a subset $E \subseteq X \times X$ is controlled with respect to a metric d on X if and only if $d|_E$ is bounded.

Example 2.1. Let G be a locally compact group and let X be a locally compact space with a continuous, proper, cocompact action of G . For instance, we may take $X = G$ with the action by left translation. There is a unique proper coarse structure on X for which any controlled subset is contained in a G -invariant controlled subset. Namely, a subset E of $X \times X$ is controlled if and only if it is contained in $E_K := \{(gx, gy) \mid g \in G, x, y \in K\}$ for some compact subset $K \subseteq X$. This is a proper coarse structure because X is locally compact, and any controlled subset is contained in a G -invariant one. Conversely, in a proper coarse structure on X with

enough G -invariant controlled sets, the sets $E_K \subseteq X \times X$ for compact $K \subseteq G$ must be controlled. And if there were more controlled subsets, some non-compact subsets would have to be bounded, which is forbidden for proper coarse structures.

If G is a finitely generated discrete group, then the coarse structure in Example 2.1 is the metric coarse structure for any word-length metric on G . If $G = \mathbb{R}^{2n}$, then the coarse structure in Example 2.1 is the metric coarse structure for the Euclidean metric on \mathbb{R}^{2n} .

The definition of a coarse structure does not carry over to noncommutative spaces. Therefore, we shall study compactifications instead of coarse structures:

Definition 2.2. Let X be a locally compact space. A *compactification* of X is a compact space \bar{X} with a homeomorphism from X onto a dense, open subset of \bar{X} .

We will explain below why compactifications are a reasonable substitute for coarse spaces. First we discuss the noncommutative version of compactifications.

Definition 2.3. Let A be a C^* -algebra. A (noncommutative) *compactification* of A is a unital C^* -algebra \bar{A} with a $*$ -isomorphism from A onto an essential ideal in \bar{A} . Being *essential* means that for any $a \in \bar{A}$ with $a \neq 0$ there is $x \in A$ with $a \cdot x \neq 0$. The quotient $\partial A := \bar{A}/A$ is called the boundary or *corona algebra* of the compactification.

The *multiplier algebra* $\mathcal{M}(A)$ of A is unital and contains A as an essential ideal, so it is a compactification. It is the largest compactification of A :

Lemma 2.4. Let $A \triangleleft \bar{A}$ be a compactification. The identity map on A extends uniquely to an isomorphism from \bar{A} onto a unital C^* -subalgebra of $\mathcal{M}(A)$ containing A . Conversely, any unital C^* -subalgebra of $\mathcal{M}(A)$ containing A gives a compactification of A . Thus we may also define a noncommutative compactification of A as a unital C^* -subalgebra of $\mathcal{M}(A)$ that contains A .

Proof. A $*$ -homomorphism $\iota: \bar{A} \rightarrow \mathcal{M}(A)$ that extends the canonical inclusion $A \triangleleft \mathcal{M}(A)$ must map $x \in \bar{A}$ to the multiplier defined by $a \mapsto x \cdot a \in A$ for $a \in A$. This indeed defines a unital $*$ -homomorphism ι , which is injective because the ideal A in \bar{A} is essential. So it identifies \bar{A} with a unital C^* -subalgebra of $\mathcal{M}(A)$ containing A . Conversely, A is an essential ideal in $\mathcal{M}(A)$ and hence in any unital C^* -subalgebra of $\mathcal{M}(A)$ that contains A . \square

Lemma 2.5. Let X be a locally compact space. There is a bijection between isomorphism classes of compactifications of X and $C_0(X)$.

Proof. Let $X \subset \bar{X}$ be a compactification of X . Then $C(\bar{X})$ is a unital C^* -algebra and extension by zero is a $*$ -isomorphism from $C_0(X)$ onto an ideal in $C(\bar{X})$. This ideal is essential because X is dense in \bar{X} . Conversely, let $C_0(X) \triangleleft A$ be a compactification. By Lemma 2.4, we may identify A with a C^* -subalgebra of $\mathcal{M}(C_0(X)) \cong C_b(X)$, so A is commutative. Thus $A \cong C(\bar{X})$ for some compact space \bar{X} . The ideal $C_0(X) \triangleleft A$ corresponds to an open subset in \bar{X} , together with a homeomorphism between this open subset and X . This open subset in \bar{X} is dense because $C_0(X) \triangleleft A$ is essential. \square

A metric gives rise to a compactification as follows:

Example 2.6. Let X be a locally compact space and let d be a continuous, proper metric on X . A bounded continuous function $f: X \rightarrow \mathbb{C}$ *oscillates slowly* if for all $R > 0$ and $\varepsilon > 0$ there is a compact subset $K \subseteq X$ such that $|f(x) - f(y)| < \varepsilon$ if $d(x, y) \leq R$ and $x, y \notin K$; the name “slowly oscillating” comes from [22]. The slowly oscillating functions form a unital C^* -subalgebra of $C_b(X) = \mathcal{M}(C_0(X))$,

which contains $C_0(X)$. Hence they provide a compactification of $C_0(X)$. The corresponding compactification of (X, d) is called *Higson compactification*.

More generally, slowly oscillating can be defined on any coarse space. So any coarse space has a (Higson) compactification.

Let $X \subseteq \bar{X}$ be a compactification of X and $\partial X := \bar{X} \setminus X$. Call a subset $E \subseteq X \times X$ *controlled* if the closure of E in $\bar{X} \times \bar{X}$ meets the set

$$\bar{X} \times \bar{X} \setminus X \times X = \bar{X} \times \partial X \cup \partial X \times \bar{X}$$

only in the diagonal; that is, if $(x_i, y_i)_{i \in I}$ is a net in E that converges in $\bar{X} \times \bar{X}$ with $\lim x_i \in \partial X$ or $\lim y_i \in \partial X$, then $\lim x_i = \lim y_i$.

Proposition 2.7. *The controlled subsets associated to a compactification $X \subseteq \bar{X}$ form a coarse structure on X whose bounded subsets are exactly the relatively compact ones. The coarse structure on X induced by \bar{X} is proper if $X \subseteq \bar{X}$ is the Higson compactification of a proper coarse structure or if X is σ -compact and ∂X is second countable. In the second case, the resulting Higson compactification is isomorphic to the original compactification \bar{X} . In general, the coarse structure defined by a compactification is proper if and only if the separated, locally closed subsets $\{(x, x) \mid x \in X\}$ and*

$$(1) \quad \{(x, y) \in \bar{X} \times \bar{X} \mid x \neq y \text{ and } x \in \partial X \text{ or } y \in \partial X\}$$

of $\bar{X} \times \bar{X}$ are separated by neighbourhoods.

Proof. All this would follow from [19, Theorem 2.27 and Proposition 2.48], but the assumptions made there do not suffice for the proof. The correction in [20] explains that the theorem works if \bar{X} is second countable. The main issue is whether there is a controlled neighbourhood of the diagonal. Lemma A.2 shows this if X is σ -compact and ∂X is second countable; then X is paracompact as assumed throughout [19]. All functions in $C(\bar{X})$ are slowly oscillating for the coarse structure defined by \bar{X} . The Higson compactification is canonically isomorphic to the original compactification \bar{X} if and only if all slowly oscillating functions belong to $C(\bar{X})$. The proof in [19] that this is so if \bar{X} is second countable goes through under our assumptions because any point in ∂X is a limit of a sequence in X by Lemma A.1.(3).

If \bar{X} is the Higson compactification of a proper coarse structure, then all controlled subsets of the original coarse structure remain controlled in the coarse structure induced by \bar{X} . Hence a controlled neighbourhood for the original coarse structure remains controlled for the new coarse structure, so that the latter is again proper.

The subset of $\bar{X} \times \bar{X}$ in (1) and $\{(x, x) \mid x \in X\}$ are separated, that is, each is disjoint from the other's closure. Being separated by neighbourhoods means that there are disjoint open subsets V and U of $\bar{X} \times \bar{X}$ that contain them. Then $U \cap X \times X$ is a controlled neighbourhood of the diagonal in $X \times X$. Conversely, a controlled neighbourhood of the diagonal in $X \times X$ contains an open controlled neighbourhood U . Thus $\bar{U} \cap \bar{X} \times \bar{X} \setminus X \times X$ is contained in the diagonal, so that the open subset $V := (\bar{X} \times \bar{X}) \setminus \bar{U}$ contains the subset in (1). So the open subsets U and V separate our two subsets. \square

Proposition 2.8 ([19, Proposition 2.47]). *Let d be a continuous proper metric on X . Build its Higson compactification as in Example 2.6. A subset $E \subseteq X \times X$ is controlled with respect to the Higson compactification if and only if $d|_E$ is bounded.*

Proposition 2.8 shows that the coarse structure and the Higson compactification associated to a proper metric on a space X contain the same information, that is, one determines the other uniquely. There are coarse structures that do not come from any compactification. But the most important examples are those defined by

metrics or by compactifications with metrisable boundary. In both cases, the coarse structure and the compactification determine each other by Propositions 2.7 and 2.8. Hence the following definition seems legitimate:

Definition 2.9. A *noncommutative coarse space* is a C^* -algebra A with a compactification $A \triangleleft \bar{A}$.

The following example shows that any unital C^* -algebra appears as a corona algebra in some noncommutative coarse space:

Example 2.10. Let B be a unital C^* -algebra. The *cone over B* is the noncommutative coarse space $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$. Its corona algebra $C(\mathbb{N}^+, B) / C_0(\mathbb{N}, B)$ is naturally isomorphic to B by evaluation at ∞ .

Definition 2.11. Let $A \triangleleft \bar{A}$ be a noncommutative coarse space. An automorphism $\alpha \in \text{Aut}(A)$ is a *translation* if its canonical extension to $\mathcal{M}(A)$ maps $\bar{A} \subseteq \mathcal{M}(A)$ to itself and induces the identity map on $\partial A := \bar{A}/A$.

Example 2.12. Let G be a locally compact group and let A be a C^* -algebra with a continuous action α of G . Let

$$\bar{A}_G := \{x \in \mathcal{M}(A) \mid \alpha_g(x) - x \in A \text{ for all } g \in G\}.$$

This is a compactification of A , and it is the largest one such that G acts by translations, that is, it acts trivially on $\partial A_G := \bar{A}_G/A$. Since G acts continuously on A and ∂A_G , it also acts continuously on \bar{A}_G by [4].

Lemma 2.13. Let $A = C_0(G)$ with the G -action by right translation. Then $\bar{A}_G \subseteq C_b(G)$ is the C^* -algebra of slowly oscillating functions for the coarse structure on G in Example 2.1.

Proof. Let $f \in C_b(G)$. By definition, f oscillates slowly for the coarse structure of Example 2.1 if and only if $|f(gx) - f(gy)| \rightarrow 0$ for $g \rightarrow \infty$, uniformly for x, y in any compact subset of G . Replacing g by gy , we see that we may as well take $y = 1$. Let $\alpha_x f(g) := f(gx)$. We may rewrite the slow oscillation condition for fixed x as $\alpha_x f - f \in C_0(G)$. Thus all slowly oscillating functions belong to $\bar{C}_0(G)_G$. For the converse, we use the automatic continuity of the G -action on \bar{A}_G , which follows from [4]. If $f \in \bar{C}_0(G)_G$, then the map $x \mapsto \alpha_x f - f$ is continuous and hence maps a compact subset of G to a compact subset K of $C_0(G) \subseteq C(G^+)$. By the Arzelà–Ascoli Theorem, the functions $\alpha_x f - f$ for $x \in K$ are uniformly continuous on the one-point compactification G^+ . Thus they vanish uniformly at infinity. This means that f oscillates slowly. \square

3. RIEFFEL DEFORMATION OF COARSE STRUCTURES

Our main examples of noncommutative coarse spaces are Rieffel deformations of commutative coarse spaces. We shall use Kasprzak’s description of Rieffel deformations in [11] because it simplifies the proof of functorial properties and is slightly more general. Mostly, we may treat Rieffel deformation as a black box and use only some functorial properties listed below. This may help to extend the theory to other situations. A candidate are the deformations in [3]; but we have not checked whether the following works in that situation.

Let G be an Abelian, locally compact group and let Ψ be a continuous 2-cocycle on G . A G - C^* -algebra is a C^* -algebra with a (strongly) continuous action of G by automorphisms. Rieffel deformation with respect to Ψ maps a G - C^* -algebra A to another G - C^* -algebra A^Ψ . This construction has the following properties:

- (RD1) Rieffel deformation is functorial for G -equivariant $*$ -homomorphisms and morphisms (nondegenerate $*$ -homomorphisms to multiplier algebras); this functor preserves injectivity of morphisms, see [11, Proposition 3.8].
- (RD2) Rieffel deformation is exact, that is, it maps an extension of G - C^* -algebras again to an extension of G - C^* -algebras, see [11, Theorem 3.9].
- (RD3) If G acts trivially on A , then $A^\Psi = A$; this is not stated in [11], but easy to prove, see Lemma 3.3 below.
- (RD4) If Ψ_1, Ψ_2 are two cocycles then $(A^{\Psi_1})^{\Psi_2} = A^{\Psi_1\Psi_2}$ with the product cocycle $\Psi_1\Psi_2$, see [11, Lemma 3.5].
- (RD5) If Ψ is the unit cocycle (constant 1), then $A^\Psi = A$; this is trivial.

Theorem 3.1. *Let $A \triangleleft \bar{A}$ be a noncommutative coarse space, equipped with a continuous action of G ; that is, G acts continuously on \bar{A} , leaving the ideal A invariant. Then $A^\Psi \triangleleft \bar{A}^\Psi$ is a noncommutative coarse space, and its corona algebra is the Rieffel deformation ∂A^Ψ .*

Proof. The ideal inclusion $A \triangleleft \bar{A}$ is equivalent to a morphism $\bar{A} \rightarrow A$, that is, to a nondegenerate $*$ -homomorphism $\bar{A} \rightarrow \mathcal{M}(A)$; this morphism is injective if and only if A is an essential ideal in \bar{A} , compare Lemma 2.4. Thus a noncommutative coarse space is the same as an extension of C^* -algebras $A \hookrightarrow \bar{A} \twoheadrightarrow \partial A$ such that \bar{A} is unital and the resulting morphism $\bar{A} \rightarrow A$ is injective.

By assumption, the extension $A \hookrightarrow \bar{A} \twoheadrightarrow \partial A$ is an extension of G - C^* -algebras and the injective morphism $\bar{A} \rightarrow A$ is G -equivariant. By (RD1) and (RD2), Rieffel deformation maps this to an extension $A^\Psi \hookrightarrow \bar{A}^\Psi \twoheadrightarrow \partial A^\Psi$, such that the resulting morphism $\bar{A}^\Psi \rightarrow A^\Psi$ is injective. That is, A^Ψ is an essential ideal in \bar{A}^Ψ , and the corona algebra \bar{A}^Ψ/A^Ψ is the Rieffel deformation ∂A^Ψ of ∂A . An algebra \bar{A} is unital if and only if there is a nondegenerate $*$ -homomorphism $\mathbb{C} \rightarrow \bar{A}$, which is equivariant for the trivial action on \mathbb{C} . Rieffel deformation maps this to a nondegenerate $*$ -homomorphism $\mathbb{C}^\Psi \rightarrow \bar{A}^\Psi$. Since $\mathbb{C}^\Psi = \mathbb{C}$ by (RD3), \bar{A}^Ψ is unital. \square

Theorem 3.2. *Let $A \triangleleft \bar{A}$ be a noncommutative coarse space and let G act on A continuously and by translations. Then G acts continuously on \bar{A} . And $A^\Psi \triangleleft \bar{A}^\Psi$ is a noncommutative coarse space with the corona algebra $\partial A^\Psi = \partial A$, and G acts on A^Ψ by translations.*

Proof. By assumption, the action on A extends to an action on \bar{A} that induces the trivial action on ∂A . Since the trivial action on ∂A and the action on A are continuous, so is the induced action on \bar{A} by the main result of [4]. Hence Rieffel deformation makes sense in this case. Theorem 3.1 shows that $A^\Psi \triangleleft \bar{A}^\Psi$ is a noncommutative coarse space with the corona algebra ∂A^Ψ . Since G acts trivially on ∂A , (RD3) gives $\partial A^\Psi = \partial A$ with the trivial action of G . So G acts on A^Ψ by translations. \square

Lemma 3.3. *Let G act trivially on A . Then $A^\Psi = A$ as subalgebras of $\mathcal{M}(A \rtimes G)$; even more, the deformed dual action on $A \rtimes G$ used by Kasprzak to define the Rieffel deformation is the same as the original dual action.*

Proof. The deformation of the dual action conjugates the automorphism by some unitary elements of $C_0(\hat{G}) \cong C^*(G) \subseteq \mathcal{M}(A \rtimes G)$. Since the action of G on A is trivial, $A \rtimes G \cong A \otimes C^*(G)$, so $C^*(G)$ is contained in the centre of the multiplier algebra. Hence conjugating by unitaries in $C^*(G)$ is the identity automorphism, and so the deformation does, in fact, not change the dual action. \square

3.4. The coarse Moyal plane. Let $G := \mathbb{R}^{2n}$ and $A := C_0(\mathbb{R}^{2n})$ with the translation action, $\alpha_g f(x) := f(x - g)$ for all $x, g \in \mathbb{R}^{2n}$. Equip G with the usual Euclidean metric and let \bar{A} be the C^* -algebra of slowly oscillating functions, see Example 2.6.

The automorphisms α_g of A are indeed “translations” as in Definition 2.11. Hence Theorem 3.2 allows us to transport this coarse structure to any Rieffel deformation of A , in such a way that the corona algebra stays the same.

If the cocycle Ψ comes from a nondegenerate antisymmetric bilinear map on \mathbb{R}^{2n} , then the Rieffel deformation for the translation action on $C_0(\mathbb{R}^{2n})$ is a Moyal plane. Its underlying C^* -algebra is isomorphic to the C^* -algebra of compact operators on the Hilbert space $L^2(\mathbb{R}^n)$. This is well-known for Rieffel’s original definition of his deformation, which is equivalent to Rieffel’s (see [14]). Thus we have found some noncommutative compactification of $\mathbb{K}(L^2\mathbb{R}^n)$, that is, a unital C^* -subalgebra \bar{A}^Ψ of $\mathbb{B}(L^2\mathbb{R}^n) \cong \mathcal{M}(\mathbb{K}(L^2\mathbb{R}^n))$, such that $\bar{A}^\Psi/\mathbb{K}(L^2\mathbb{R}^n)$, its image in the Calkin algebra, is commutative. Namely, $\bar{A}^\Psi/\mathbb{K}(L^2\mathbb{R}^n) \cong C(\partial\mathbb{R}^{2n})$ is the C^* -algebra of continuous functions on the Higson corona of \mathbb{R}^{2n} for the Euclidean metric.

Next we describe this compactification of the Moyal plane directly, without any reference to the classical Euclidean plane. The group \mathbb{R}^{2n} acts on $L^2\mathbb{R}^n$ by a projective representation, where one copy of \mathbb{R}^n acts by translation, the other by pointwise multiplication with characters. This projective representation on $L^2\mathbb{R}^n$ induces a strongly continuous \mathbb{R}^{2n} -action β on $B := \mathbb{K}(L^2\mathbb{R}^n)$. This is equivalent to the \mathbb{R}^{2n} -action on $C_0(\mathbb{R}^{2n})^\Psi \cong \mathbb{K}(L^2\mathbb{R}^n)$ as a Rieffel deformation.

Theorem 3.5. *The coarse structure on the Moyal plane $B := \mathbb{K}(L^2\mathbb{R}^n)$ constructed by Rieffel deformation of the classical Euclidean plane is*

$$\bar{B} := \{x \in \mathbb{B}(L^2\mathbb{R}^n) \mid \beta_g(x) - x \in \mathbb{K}(L^2\mathbb{R}^n) \text{ for all } g \in \mathbb{R}^{2n}\},$$

the largest compactification for which the \mathbb{R}^{2n} -action β is by translations; see also Example 2.12.

Proof. Let $B \triangleleft \bar{B}$ be the coarse structure constructed in Example 2.12. It is the largest coarse structure such that \mathbb{R}^{2n} acts by translations. Let Ψ be the cocycle on \mathbb{R}^{2n} such that $B = A^\Psi$ with $A = C_0(\mathbb{R}^{2n})$. Thus $B^{\Psi^*} = A = C_0(\mathbb{R}^{2n})$ by (RD4) and (RD5). By Theorem 3.2, the action of \mathbb{R}^{2n} on A is by translations with respect to the compactification \bar{B}^{Ψ^*} . Even more, we claim that \bar{B}^{Ψ^*} is the largest compactification of A for which \mathbb{R}^{2n} acts by translations. Indeed, if \bar{A} is such a compactification of A , then \mathbb{R}^{2n} still acts by translations on \bar{A}^Ψ by Theorem 3.2, so that $\bar{A}^\Psi \subseteq \bar{B}$; and then $\bar{A} = \bar{A}^{\Psi\Psi^*} = (\bar{A}^\Psi)^{\Psi^*} \subseteq \bar{B}^{\Psi^*}$ by (RD5), (RD4) and (RD1). The largest compactification of \mathbb{R}^{2n} where \mathbb{R}^{2n} acts by translations is the one described in Example 2.12. Lemma 2.13 identifies it with the standard coarse structure on \mathbb{R}^{2n} , which is the same as the coarse structure from the Euclidean metric on \mathbb{R}^{2n} . So $\bar{B} = (\bar{B}^{\Psi^*})^\Psi \cong \bar{A}^\Psi$ is the Rieffel deformation of the Higson compactification of $C_0(\mathbb{R}^{2n})$ for the standard coarse structure on \mathbb{R}^{2n} . \square

Now let $\bar{\mathbb{R}^{2n}}$ be the usual ball compactification with the boundary $\partial\mathbb{R}^{2n} \cong \mathbb{S}^{2n-1}$. All continuous functions on $\bar{\mathbb{R}^{2n}}$ oscillate slowly with respect to the Euclidean metric on \mathbb{R}^{2n} . When we apply Rieffel deformation as above, then we get a compactification of the Moyal plane $\mathbb{K}(L^2\mathbb{R})$ by $C(\mathbb{S}^{2n-1})$. The Rieffel deformation of $C(\bar{\mathbb{R}^{2n}})$ is already studied by Coburn and Xia [6]. They identify it with the Toeplitz algebra of the unit ball in \mathbb{C}^n . In particular, for $n = 1$ we get the usual Toeplitz algebra, the universal C^* -algebra generated by a single isometry. Since $C(\bar{\mathbb{R}^{2n}})$ is contained in the Higson compactification for the Euclidean metric, their Rieffel deformations are also contained in one another. So the compactification described in Theorem 3.5 may be related to Toeplitz operators with discontinuous symbols.

4. COCOMPACT CONTINUOUSLY SQUARE-INTEGRABLE GROUP ACTIONS

Example 2.1 describes a unique “ G -invariant” coarse structure on a locally compact space X with a continuous, proper, cocompact action of a locally compact

group G . The space X equipped with this coarse structure is coarsely equivalent to G with the standard coarse structure: the orbit inclusions $G \rightarrow X$, $g \mapsto g \cdot x$, for $x \in X$ are coarse equivalences. We are going to extend this construction to the noncommutative case. So let A be a C^* -algebra with a continuous action α of G . We first have to define when α is “proper” and “cocompact.” We interpret “proper” as *continuously square-integrable* as in [13], see also [16, 18].

A continuously square-integrable action comes with some additional data. We shall use the equivalent characterisation of continuous square-integrability in [13, Theorem 6.1] (in the case of the Hilbert A -module $\mathcal{E} = A$): a continuously square-integrable structure for (A, α, G) is equivalent to a Hilbert module \mathcal{F} over $A \rtimes_{r, \alpha} G$ with a G -equivariant unitary of Hilbert A -modules

$$(2) \quad U: A \xrightarrow{\sim} \mathcal{F} \otimes_{A \rtimes_{r, \alpha} G} L^2(G, A);$$

here we use the faithful representation of the reduced crossed product $A \rtimes_{r, \alpha} G$ on the Hilbert A -module $L^2(G, A)$ by G -equivariant operators described in [13, §3]. Let $\mathcal{M}(A)^G$ be the C^* -algebra of G -invariant multipliers of A . The *generalised fixed point algebra* of the continuously square-integrable action with Hilbert module \mathcal{F} is defined to be $\mathbb{K}(\mathcal{F})$, viewed as a C^* -subalgebra of $\mathcal{M}(A)^G$ through the representation

$$\mathbb{K}(\mathcal{F}) \xrightarrow{T \mapsto T \otimes 1} \mathbb{B}(\mathcal{F} \otimes_{A \rtimes_{r, \alpha} G} L^2(G, A))^G \xrightarrow[\cong]{\text{Ad } U^*} \mathbb{B}(A)^G = \mathcal{M}(A)^G.$$

This contains the construction of a generalised fixed point algebra by Rieffel [16] as a special case.

Let X be a locally compact space with a continuous action of G . Let $A = C_0(X)$ and let α be the induced action of G on A , which is continuous. As shown in [13, §9], this action is continuously square-integrable if and only if the action on G is proper; the Hilbert module \mathcal{F} is unique in this case; and the generalised fixed point algebra is $C_0(X/G)$. This is unital if and only if X/G is compact. This justifies the following definition:

Definition 4.1. A continuously square-integrable action is *cocompact* if its generalised fixed point algebra is unital.

From now on, we assume the action α to be cocompact continuously square-integrable. Implicitly, this fixes a Hilbert module \mathcal{F} over $B := A \rtimes_{r, \alpha} G$ and a G -equivariant unitary Hilbert A -module isomorphism U as in (2). In this situation, we want to define a coarse structure on A .

Proposition 4.2. *Let B be a C^* -algebra and \mathcal{F} a Hilbert B -module. The C^* -algebra $\mathbb{K}(\mathcal{F})$ is unital if and only if there are $n \in \mathbb{N}$, a projection $p \in \mathbb{M}_n(B)$, and a unitary $\mathcal{F} \cong p \cdot B^n$. The projection p is unique up to Murray–von Neumann equivalence.*

Proof. The second claim is trivial. The first one is contained in well known results if B is unital. See, for instance, [21, Theorem 15.4.2] and the remarks following it. If B is not unital, then it still follows by similar arguments. If $\mathcal{F} \cong p \cdot B^n$ for a projection $p \in \mathbb{M}_n(B)$, then $\text{id}_{\mathcal{F}}$ is compact because so is p and compact operators form an ideal. Conversely, assume that $\text{id}_{\mathcal{F}}$ is compact. Then $\mathbb{K}(\mathcal{F})$ is σ -unital and hence \mathcal{F} is countably generated. So $\mathcal{F} \cong p_1 \cdot \ell^2(\mathbb{N}, B)$ for a projection $p_1 \in \mathbb{B}(\ell^2(\mathbb{N}, B))$ by the Kasparov Stabilisation Theorem. Actually, $p_1 \in \mathbb{K}(\ell^2(\mathbb{N}, B))$ because $\text{id}_{\mathcal{F}}$ is compact. Since finite matrices are dense in the compact operators, there are $n \in \mathbb{N}$ and $p_2 \in \mathbb{M}_n(B)$ so that $\|p_2 - p_1\| < 1/2$. Then p_2 is close to a projection $p \in \mathbb{M}_n(B)$ that is Murray–von Neumann equivalent to p_1 . Thus $p \cdot B^n \cong \mathcal{F}$. \square

Fix a projection $p \in \mathbb{M}_n(B)$ as in Proposition 4.2. Using the unitary U in (2), we identify

$$A \cong \mathcal{F} \otimes_B L^2(G, A) \cong p \cdot B^n \otimes_B L^2(G, A) \cong p \cdot L^2(G, A^n)$$

for the representation of $\mathbb{M}_n(B)$ on $L^2(G, A^n)$ induced by the standard representation of B on $L^2(G, A)$ by G -equivariant operators. Let $C(\eta G) \subseteq C_b(G)$ be the Higson compactification for the coarse structure on G in Example 2.1. This is equal to the compactification of $C_0(G)$ described in Example 2.12 by Lemma 2.13. Let M denote the representation of $C(\eta G)$ on $L^2(G, A) \otimes \mathbb{C}^n \cong L^2(G, A^n)$ by pointwise multiplication in the L^2G -direction. Let

$$\varphi: C(\eta G) \rightarrow \mathbb{B}(A), \quad f \mapsto pM(f)p,$$

the compression of the representation by pointwise multiplication to the direct summand $p \cdot L^2(G, A^n) \cong A$.

Theorem 4.3. *The subspace $\bar{A} := A + \varphi(C(\eta G)) \subseteq \mathcal{M}(A)$ is a compactification of A with boundary $\bar{A}/A \cong C(\partial\eta G)$.*

Proof. Consider the dense subalgebras $C_c(G) \subseteq C_0(G)$ and $C_c(G) \subseteq C_r^*(G)$. When we represent both on $L^2(G)$, then $M(C_c(G)) \cdot \varrho(C_c(G))$ is dense in the algebra of compactly supported integral kernels. Hence $C_0(G) \cdot C_r^*(G) = \mathbb{K}(L^2G)$. Similarly, when we represent $C_0(G)$ and $A \rtimes_{r,\alpha} G$ on $L^2(G, A)$ as above, then

$$C_0(G) \cdot (A \rtimes_{r,\alpha} G) = \mathbb{K}(L^2(G, A)).$$

This is equivalent to the Imai–Takai Duality Theorem for crossed products for group actions and group coactions, which asserts an isomorphism $(A \rtimes_r G) \rtimes_{\hat{G}} \cong A \otimes \mathbb{K}(L^2G)$ (see [9] or [8, Theorem A.69]).

The images of $C(\eta G)$ and $\mathbb{M}_n(A)$ in $\mathbb{B}(L^2(G, A^n))$ commute. And $u_g f u_g^* - f = \lambda_g(f) - f \in C_0(G)$ for all $f \in C(\eta G)$, $g \in G$ by the definition of ηG . Therefore, $[B, C(\eta G)] \subseteq C_0(G) \cdot B = \mathbb{K}(L^2(G, A))$ as well. So the map φ above is multiplicative modulo $\mathbb{K}(A) = A$. Since $\varphi(x^*) = \varphi(x)^*$, the induced map $C(\eta G) \rightarrow \mathcal{M}(A)/A$ is a $*$ -homomorphism. Thus its image is a C^* -subalgebra of $\mathcal{M}(A)/A$. So $A + \varphi(C(\eta G))$ is a C^* -subalgebra of $\mathcal{M}(A)$. It is unital because $\varphi(1) = p$ is the identity operator on $A \cong p \cdot L^2(G, A^n)$. The argument above also shows that $\varphi(C_0(G)) \subseteq A$. Thus the boundary \bar{A}/A is a quotient of $C(\eta G)/C_0(G) = C(\partial\eta G)$. To show that the induced map $C(\eta G)/C_0(G) \rightarrow \bar{A}/A$ is an isomorphism, we sketch the construction of a map $\chi: \bar{A} \rightarrow C(\eta G)$ with $\chi(A) \subseteq C_0(G)$ and so that $\chi \circ \varphi$ induces the identity map on $C(\eta G)/C_0(G)$.

Let $w \in C_c(G)$ be a positive function supported in a small neighbourhood of 1 with $\int_G w^2(g) dg = 1$. Let $w^g(x) := w(g^{-1}x)$ for $g, x \in G$. We map

$$\chi_1^g: \mathbb{B}(A) \cong \mathbb{B}(p \cdot L^2(G, A^n)) \rightarrow \mathbb{B}(L^2(G, A^n)) \xrightarrow{C_{w^g}} \mathbb{B}(A^n) = \mathbb{M}_n(\mathcal{M}(A)),$$

where the first arrow extends an operator by 0 on the orthogonal complement and C_{w^g} takes the matrix coefficient of $w^g \in L^2(G)$, that is, $\langle C_{w^g}(x)\xi, \eta \rangle = \langle x(\xi \otimes w^g), \eta \otimes w^g \rangle$ for all $\xi, \eta \in A^n$, $x \in \mathbb{B}(L^2(G, A^n))$, $g \in G$. The operator $\chi_1^1(1) = C_w(p)$ is positive. If this is 0, then p vanishes on $w \otimes A^n$ and hence on $g \cdot w \otimes A^n$ for all $g \in G$. Since the G -orbits of functions of the form w are dense in L^2G and $p \neq 0$, there must be $w \in C_c(G)$ as above with $C_w(p) \neq 0$. Then we can find a positive linear functional $\sigma \in A'$ with $\sigma(C_w(p)) = 1$ and $\|\sigma\| \|C_w(p)\| = 1$.

Recall that $p \in \mathbb{M}_n(A) \rtimes_r G$ acts on $L^2(G, A^n)$ by a G -equivariant operator, that is, p commutes with the operators

$$U_g: L^2(G, A^n) \rightarrow L^2(G, A^n), \quad (U_g f)(x) := \alpha_g(f(g^{-1}x)).$$

Thus $C_w(p) = C_w(U_g^{-1}pU_g) = \alpha_g(C_{w^g}(p))$ for all $g \in G$. We define

$$\chi: \mathcal{M}(A) = \mathbb{B}(A) \rightarrow \mathbb{C}^G, \quad \chi(x)(g) := \sigma \circ \alpha_g(\chi_1^g(x)).$$

The map χ is completely positive by construction, and

$$\chi(1)(g) = \sigma \alpha_g(C_{w^g}(p)) = \sigma(C_w(p)) = 1$$

for all $g \in G$. Thus the function $\chi(x)$ on G is bounded for all $x \in \mathcal{M}(A)$ and $\|\chi\| = 1$ as a map $\mathcal{M}(A) \rightarrow \mathbb{C}^G$.

If $x \in A = \mathbb{K}(A)$, then its image in $\mathbb{B}(L^2(G, A^n))$ is compact, so we may approximate it by compactly supported integral kernels $G \times G \rightarrow \mathbb{M}_n(A)$. For such an integral kernel k , $g \mapsto \sigma(C_{w^g}(k))$ is continuous of compact support. Hence $\chi(A) \subseteq C_0(G)$ as desired. Let $f \in C(\eta G)$. Then

$$\begin{aligned} \chi(\varphi(f))(g) &= \sigma \alpha_g(C_{w^g}(pM(f)p)) \\ &= \sigma(C_w(U_g^{-1}pM(f)pU_g)) = \sigma(C_w(pM(\lambda_{g^{-1}}f)p)). \end{aligned}$$

Approximate p by some $\tilde{p} \in C_c(G, \mathbb{M}_n(A))$ supported in a compact subset $K \subseteq G$. Then $C_w(\tilde{p}M(\lambda_{g^{-1}}f)\tilde{p})$ only involves the values of $\lambda_{g^{-1}}f$ on $\text{supp } w \cdot K$, that is, the values of f on $g \cdot \text{supp } w \cdot K$. Since $f \in C(\eta G)$, $f|_{g \cdot K \cdot \text{supp } w}$ gets more and more constant. Therefore,

$$0 = \lim_{g \rightarrow \infty} \chi(\varphi(f))(g) - f(g)\sigma(C_w(p)) = \lim_{g \rightarrow \infty} \chi(\varphi(f))(g) - f(g).$$

Furthermore, since the G -action on $C(\eta G)$ is continuous, $\chi(\varphi(f))$ is a continuous function on G . Thus $\chi(\varphi(f)) - f \in C_0(G)$. So $\chi(\bar{A}) \subseteq C(\eta G)$ and $\chi \circ \varphi$ induces the identity map on $C(\eta G)/C_0(G)$. \square

5. NONCOMMUTATIVE COARSE MAPS AND EQUIVALENCES

We have encountered two situations that smell of a coarse equivalence between two noncommutative spaces. The first one is Rieffel deformation for an action of an Abelian locally compact group G by translations, where we expect a coarse equivalence between $A \triangleleft \bar{A}$ and $A^\Psi \triangleleft \bar{A}^\Psi$. The second is the coarse structure for a cocompact continuously square-integrable action in Theorem 4.3, which we expect to be coarsely equivalent to $C_0(G)$ with its usual coarse structure. How should we define coarse maps between noncommutative coarse spaces, so as to cover these two situations? In both cases, the corona algebras are isomorphic. But a *-homomorphism between the boundaries is a poor definition of a coarse map.

The example of the Moyal plane as a Rieffel deformation of $C_0(\mathbb{R}^{2n})$ shows that we cannot expect a *-homomorphism between A and A^Ψ . This is not surprising because already in the commutative case, we need discontinuous coarse maps, say, for the coarse equivalence between \mathbb{R} and \mathbb{Z} . An obvious way to allow for “discontinuous” *-homomorphisms would be to replace a C^* -algebra by its bidual W^* -algebra. This contains the C^* -algebra of Borel functions on X for $C_0(X)$, so that a normal *-homomorphism between the biduals would allow for a Borel map between the locally compact spaces. This idea fails, however, for the Moyal plane: its underlying C^* -algebra is that of compact operators, so its bidual is the W^* -algebra $\mathbb{B}(\mathcal{H})$ of bounded operators on a Hilbert space. There are no normal *-homomorphisms from this to a commutative W^* -algebra.

Rieffel’s definition of his deformation is based on an isomorphism between the Fréchet subspaces of smooth elements in A and A^Ψ for the actions of G . Once again, such a densely defined, unbounded map seems a poor way to define noncommutative coarse maps. Another “quantisation map” between A and A^Ψ constructed in [10] has much better properties. These were the motivation for the following definition of a noncommutative coarse map. As we shall see, the coarse structure for a

cocompact continuously square-integrable group action in Theorem 4.3 also comes with noncommutative coarse maps in this sense by construction.

Definition 5.1. A *noncommutative coarse map* between two noncommutative coarse spaces $A \triangleleft \bar{A}$ and $B \triangleleft \bar{B}$ is a commuting diagram of maps

$$\begin{array}{ccccc} A & \hookrightarrow & \bar{A} & \twoheadrightarrow & \partial A \\ \downarrow \varphi & & \downarrow \bar{\varphi} & & \downarrow \partial \varphi \\ B & \hookrightarrow & \bar{B} & \twoheadrightarrow & \partial B, \end{array}$$

with the following properties:

- (1) $\bar{\varphi}$ is a unital, completely positive map that is strictly continuous, that is, a continuous map if both \bar{A} and \bar{B} carry the strict topology;
- (2) $\partial \varphi$ is a $*$ -homomorphism.

Since $\bar{\varphi}$ is strictly continuous and completely positive, so is φ . Even more, if $(u_i)_{i \in I}$ is an approximate unit in A , then the net $(u_i)_{i \in I}$ converges strictly to $1 \in \bar{A}$ and hence $\varphi(u_i)_{i \in I}$ converges strictly to $\bar{\varphi}(1) = 1$ in $\mathcal{M}(B)$. Thus φ is a *nondegenerate completely positive map*. Conversely, let $\varphi: A \rightarrow B$ be a nondegenerate completely positive map; that is, φ is completely positive and the net $\varphi(u_i)_{i \in I}$ converges strictly to 1 if $(u_i)_{i \in I}$ is an approximate unit in A . Then φ extends uniquely to a strictly continuous, unital, completely positive map $\mathcal{M}(\varphi): \mathcal{M}(A) \rightarrow \mathcal{M}(B)$; this follows, say, from the Stinespring Dilation Theorem for nondegenerate completely positive maps, see [12, Corollary 5.7]. (Lance only asserts strict continuity on the unit balls. The Cohen–Hewitt Factorisation Theorem allows to prove strict continuity everywhere.)

The arguments above show that a noncommutative coarse map is already determined by the map $\varphi: A \rightarrow B$. This is a nondegenerate, completely positive map $\varphi: A \rightarrow B$ with the following two extra properties:

- $\mathcal{M}(\varphi): \mathcal{M}(A) \rightarrow \mathcal{M}(B)$ maps $\bar{A} \subseteq \mathcal{M}(A)$ to $\bar{B} \subseteq \mathcal{M}(B)$, giving $\bar{\varphi}: \bar{A} \rightarrow \bar{B}$;
- $\bar{\varphi}(a_1 a_2) - \bar{\varphi}(a_1) \bar{\varphi}(a_2) \in B$ for all $a_1, a_2 \in \bar{A}$.

The second condition says that the induced map $\partial \varphi: \partial A \rightarrow \partial B$ is multiplicative; this map exists because $\bar{\varphi}(\bar{A}) \subseteq \bar{B}$ and $\bar{\varphi}(A) \subseteq B$. Then $\partial \varphi$ is a $*$ -homomorphism because, as a completely positive map, it is linear and preserves adjoints. Any nondegenerate, completely positive map $\varphi: A \rightarrow B$ with these two properties gives a noncommutative coarse map.

To define coarse equivalences, we must also define when two noncommutative coarse maps are “close.” In the commutative case, closeness may be defined for maps from any set into a coarse space. We define closeness in similar generality:

Definition 5.2. Let $A \triangleleft \bar{A}$ be a noncommutative coarse space and let B be a C^* -algebra. Two nondegenerate completely positive maps $\varphi, \psi: A \rightrightarrows B$ are *close* if their strictly continuous extensions $\mathcal{M}(\varphi), \mathcal{M}(\psi): \mathcal{M}(A) \rightrightarrows \mathcal{M}(B)$ satisfy $(\mathcal{M}(\varphi) - \mathcal{M}(\psi))(\bar{A}) \subseteq B$. Thus two noncommutative coarse maps $(\varphi, \bar{\varphi}, \partial \varphi)$ and $(\psi, \bar{\psi}, \partial \psi)$ from $A \triangleleft \bar{A}$ to $B \triangleleft \bar{B}$ are close if and only if $\partial \varphi = \partial \psi$.

Noncommutative coarse maps may be composed in an obvious fashion. So they define a category of noncommutative coarse spaces. This composition respects the closeness relation. So equivalence classes of noncommutative coarse maps up to closeness still form a category. Taking the boundary quotient is a functor in this quotient category. A noncommutative coarse equivalence is defined as an isomorphism in this quotient category. More explicitly:

Definition 5.3. A noncommutative coarse map φ is called a *coarse equivalence* if there is a noncommutative coarse map ψ in the opposite direction such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are close to the identity maps.

Example 5.4. Let A carry a cocompact continuously square-integrable action of a locally compact group G . The maps $\varphi: C_0(G) \rightarrow A$ and $\chi: A \rightarrow C_0(G)$ constructed before Theorem 4.3 and during its proof are noncommutative coarse maps that are inverse to each other up to closeness. So they form a coarse equivalence between A and $C_0(G)$. This is expected because in the commutative case, the unique G -invariant coarse structure on a cocompact proper G -space is coarsely equivalent to G . We have already constructed φ and χ as unital, completely positive maps between \bar{A} and $C(\eta G)$. It is routine to check that φ and χ are strictly continuous, using the known fact that all states on A are strictly continuous. We checked during the proof of Theorem 4.3 that φ induces a $*$ -homomorphism between the boundaries; this is an isomorphism by construction of \bar{A} . And we checked that χ induces the inverse map φ^{-1} between the boundaries, so $\partial\chi$ is a $*$ -homomorphism as well.

To further justify our definition of a noncommutative coarse map, we now construct them in the context of Rieffel deformations. So let G be a locally compact Abelian group and let Ψ be a 2-cocycle on G . Let $A \triangleleft \bar{A}$ be a noncommutative coarse space with a G -action α by translations. We want to construct a noncommutative coarse map $(\varphi, \bar{\varphi}, \partial\varphi)$ from $A \triangleleft \bar{A}$ to $A^\Psi \triangleleft \bar{A}^\Psi$ such that $\partial\varphi$ is the canonical isomorphism $\partial A \cong \partial A^\Psi$ in Theorem 3.2.

The following construction works for an arbitrary G - C^* -algebra A , so it also applies to \bar{A} and ∂A . We need Kasprzak's description of the Rieffel deformation A^Ψ and recall some of his notation, see [11]. Let $B := A \rtimes_\alpha G$ and let $\hat{\alpha}: \hat{G} \rightarrow \text{Aut}(B)$ be the dual action on the crossed product. Let $\lambda: C_r^*(G) \rightarrow \mathcal{M}(B)$ be the standard inclusion. The triple $(B, \hat{\alpha}, \lambda)$ is called a G -product. There is also a canonical embedding $A \subseteq \mathcal{M}(B)$, and its image may be described using the G -product structure: it consists of those elements of $\mathcal{M}(B)$ that satisfy *Landstad's conditions*. In order to deform A , Kasprzak deforms the dual action $\hat{\alpha}$ using the 2-cocycle Ψ . The new dual action $\hat{\alpha}^\Psi$ is still part of a G -product $(B, \hat{\alpha}^\Psi, \lambda)$, and A^Ψ is the resulting Landstad algebra. Let $b \in B$ and $f_1, f_2 \in C_r^*(G) \cap L^2(G)$. Then

$$E^\Psi(f_1 b f_2) := \int_{\hat{G}} \hat{\alpha}_\gamma^\Psi(f_1 b f_2) d\gamma$$

belongs to A^Ψ , and $\|E^\Psi(f_1 b f_2)\| \leq \|f_1\|_2 \|b\| \|f_2\|_2$; here the integral converges in the strict topology on $\mathcal{M}(B)$. Furthermore, Kasprzak shows that elements of this form are dense in A^Ψ .

We want to use the same formula for $b \in A \subseteq \mathcal{M}(B)$, so b no longer belongs to B . But then $f \cdot b \in B$ and $b \cdot f \in B$ for any $f \in C_r^*(G)$. The $C_r^*(G)$ -module $C_r^*(G) \cap L^2(G)$ with the norm $f \mapsto \|f\|_{C_r^*(G)} + \|f\|_2$ is nondegenerate because $C_c(G)$ is dense. By the Cohen–Hewitt Factorisation Theorem, any $f_1 \in C_r^*(G) \cap L^2(G)$ may also be written as $f_1 = f'_1 \cdot f''_1$ with $f'_1 \in C_r^*(G) \cap L^2(G)$, $f''_1 \in C_r^*(G)$, and

$$\|f'_1\|_{C_r^*(G) \cap L^2(G)} \|f''_1\|_{C_r^*(G)} \leq \|f_1\|_{C_r^*(G) \cap L^2(G)} + \varepsilon$$

for any $\varepsilon > 0$. Since $C_r^*(G) \cdot A \subseteq B$, we get $E^\Psi(f_1 a f_2) \in A^\Psi$ and

$$\|E^\Psi(f_1 a f_2)\| \leq \|f_1\|_2 \|a\| \|f_2\|_2$$

if $f_1, f_2 \in C_r^*(G) \cap L^2(G)$, $a \in A$.

Theorem 5.5. Let G be a locally compact Abelian group with a 2-cocycle Ψ . Let $A \triangleleft \bar{A}$ be a noncommutative coarse space with a continuous G -action by translations. Let $f \in L^2(G) \cap C_r^*(G)$ satisfy $\|f\|_2 = 1$. Then $\bar{\varphi}(a) := E^\Psi(f a f^*)$ for $a \in \bar{A}$ and the same formula for $\varphi = \bar{\varphi}|_A$ and $\partial\varphi$ define a noncommutative coarse map from

$A \triangleleft \bar{A}$ to $A^\Psi \triangleleft \bar{A}^\Psi$. The map $\partial\varphi$ is the canonical isomorphism $\partial A = \partial A^\Psi$ for any f . All these maps for different f are close.

Proof. The naturality of the construction of φ gives a commuting diagram as in Definition 5.1. A map of the form $a \mapsto faf^*$ is always completely positive. Since automorphisms are $*$ -homomorphisms, they are completely positive, and so is an integral over a family of completely positive maps. Thus $a \mapsto \int_K \hat{\alpha}_\gamma^\Psi(faf^*) d\gamma$ is a completely positive map $A \mapsto \mathcal{M}(B)$ for any compact subset $K \subseteq \hat{G}$. Finally, we let $K \rightarrow \hat{G}$ and take a strict limit. This still gives a completely positive map. So $a \mapsto E^\Psi(faf^*)$ is a completely positive map $A \rightarrow A^\Psi$ for any $f \in C_r^*(G) \cap L^2(G)$. If $a_2 \in A^\Psi$, then $\hat{\alpha}_\gamma^\Psi(a_2) = a_2$ for all $\gamma \in \hat{G}$. So

$$a_2 \cdot E^\Psi(faf^*) = E^\Psi(a_2 \cdot faf^*), \quad E^\Psi(faf^*) \cdot a_2 = E^\Psi(faf^* \cdot a_2).$$

The products $a_2 f$ and $f^* a_2$ belong to $C_r^*(G, A) \cap L^2(G, A)$, on which A acts nondegenerately. So $a_2 f a$ and $a f^* a_2$ go to 0 in the norm of $C_r^*(G, A) \cap L^2(G, A)$ if a goes to 0 strictly. This suffices for $a_2 \cdot E^\Psi(faf^*)$ and $E^\Psi(faf^*) \cdot a_2$ to go to 0 in norm if a goes to 0 strictly. That is, the map $a \mapsto E^\Psi(faf^*)$ is strictly continuous.

Since the G -action on ∂A is trivial, ∂A and $C_r^*(G)$ commute in B ; so

$$E^\Psi(faf^*) = a \cdot \int_{\hat{G}} \hat{\alpha}_\gamma^\Psi(f * f^*) d\gamma$$

in ∂A^Ψ for all $a \in \partial A$. The integral above is a constant multiple of the identity element, where the constant is $|\hat{f}(1)|^2 = \|f\|_2^2 = 1$ by the normalisation assumption. So the map $a \mapsto E^\Psi(faf^*)$ is the identity map from $\partial A \subseteq \mathcal{M}(B)$ to $\partial A^\Psi \subseteq \mathcal{M}(B)$, which is a $*$ -homomorphism. Since it does not depend on f , the maps for different f are all close to each other. The same computation ensures that $\bar{\varphi}(1) = 1$. \square

Proposition 5.6. *Let $A \triangleleft \bar{A}$ be a noncommutative coarse space with a continuous G -action by translations. Let Ψ_1, Ψ_2 be continuous 2-cocycles on G . Define noncommutative coarse maps*

$$\begin{aligned} (\varphi_{01}, \bar{\varphi}_{01}, \partial\varphi_{01}) &: (A \triangleleft \bar{A}) \rightarrow (A^{\Psi_1} \triangleleft \bar{A}^{\Psi_1}), \\ (\varphi_{12}, \bar{\varphi}_{12}, \partial\varphi_{12}) &: (A^{\Psi_1} \triangleleft \bar{A}^{\Psi_1}) \rightarrow ((A^{\Psi_1})^{\Psi_2} \triangleleft (\bar{A}^{\Psi_1})^{\Psi_2}), \\ (\varphi_{02}, \bar{\varphi}_{02}, \partial\varphi_{02}) &: (A \triangleleft \bar{A}) \rightarrow (A^{\Psi_1+\Psi_2} \triangleleft \bar{A}^{\Psi_1+\Psi_2}). \end{aligned}$$

Identify $A^{\Psi_1+\Psi_2} \cong (A^{\Psi_1})^{\Psi_2}$ and $(\bar{A}^{\Psi_1})^{\Psi_2} \cong \bar{A}^{\Psi_1+\Psi_2}$ as in (RD4). Then φ_{02} is close to $\varphi_{12} \circ \varphi_{01}$.

Proof. Both φ_{02} and $\varphi_{12} \circ \varphi_{01}$ induce the canonical isomorphism on the corona algebras, hence they are close. \square

Theorem 5.7. *The coarse map from $A \triangleleft \bar{A}$ to $A^\Psi \triangleleft \bar{A}^\Psi$ constructed in Theorem 5.5 is a coarse equivalence.*

Proof. Apply Proposition 5.6 with $\Psi_1 = \Psi$ and $\Psi_2 = \Psi^*$ and use that the noncommutative coarse maps from $A \triangleleft \bar{A}$ to itself constructed in Theorem 5.5 are close to the identity map because they induce the identity map on the corona algebras. \square

6. COARSE MAPS: THE COMMUTATIVE CASE

How are our noncommutative coarse maps related to coarse maps between ordinary coarse spaces? As we shall see, ordinary coarse maps give noncommutative coarse maps. The converse holds for coarse structures from compactifications with second countable boundary. It is unclear, however, for coarse structures defined by metrics. For commutative C^* -algebras, we are working with compactifications and

not with coarse structures. Hence we first have to discuss two categories of spaces with different extra structure.

Definition 6.1. Let \mathcal{C} be the category of proper coarse spaces with coarse maps, that is, maps that are proper and preserve controlled subsets. Let \mathcal{S} be the category of compactified spaces $X \subseteq \bar{X}$, where X is a σ -compact, locally compact space with a compactification \bar{X} ; the morphisms in \mathcal{S} from $X \subseteq \bar{X}$ to $Y \subseteq \bar{Y}$ are maps $\bar{X} \rightarrow \bar{Y}$ that map X into Y and ∂X into ∂Y , and that are continuous on ∂X . Two morphisms in \mathcal{S} are called *close* if they induce the same map $\partial X \rightarrow \partial Y$.

Let $X \subseteq \bar{X}$ and $Y \subseteq \bar{Y}$ be objects of \mathcal{S} and let $\bar{F}: \bar{X} \rightarrow \bar{Y}$ be a morphism between them. Then F restricts to a map $F: X \rightarrow Y$ and to a continuous map $\partial F: \partial X \rightarrow \partial Y$. We claim that F determines \bar{F} uniquely. Let $x \in \partial X$. Since X is dense in \bar{X} , there is a net $(x_i)_{i \in I}$ in X that converges towards x in \bar{X} . Since \bar{F} is continuous at x , $\bar{F}(x) = \lim F(x_i)$.

Thus we may also define the morphisms in \mathcal{S} as maps $F: X \rightarrow Y$ that have an extension $\bar{F}: \bar{X} \rightarrow \bar{Y}$ that is continuous on ∂X and that maps ∂X to ∂Y . The latter condition means that F is proper, that is, preimages of relatively compact subsets of Y are relatively compact in X . The map F need not be continuous, just as coarse maps are not required to be continuous.

Proposition 6.2 ([19, Proposition 2.41]). *The Higson compactification is part of a functor $H: \mathcal{C} \rightarrow \mathcal{S}$, that is, a coarse map $X \rightarrow Y$ has an extension $\bar{X} \rightarrow \bar{Y}$ that is continuous on ∂X and maps ∂X to ∂Y . This functor preserves closeness of morphisms.*

Conversely, a compactification \bar{X} of a σ -compact, locally compact space X induces a coarse structure $T(\bar{X})$ on X . Let $\mathcal{S}_p \subseteq \mathcal{S}$ be the full subcategory consisting of those compactified spaces where the subset in (1) and $\{(x, x) \mid x \in X\}$ in $\bar{X} \times \bar{X}$ are separated by neighbourhoods. This is equivalent to $T(\bar{X})$ being proper, see Proposition 2.7. It is also observed there that $H(\mathcal{C}) \subseteq \mathcal{S}_p$.

Lemma 6.3. *A morphism \bar{F} from $X \subseteq \bar{X}$ to $Y \subseteq \bar{Y}$ in \mathcal{S}_p restricts to a coarse map $X \rightarrow Y$ with respect to the coarse structures $T(\bar{X})$ and $T(\bar{Y})$. Thus we get a functor $T: \mathcal{S}_p \rightarrow \mathcal{C}$. It preserves closeness of morphisms.*

Proof. Continuity of \bar{F} at $x \in \partial X$ means that $\bar{F}^{-1}(U)$ is a neighbourhood of x if U is a neighbourhood of $\bar{F}(x)$. Since \bar{F} is continuous on ∂X , $\bar{F}^{-1}(U)$ is a neighbourhood of ∂X if U is a neighbourhood of $\partial Y \supseteq \bar{F}(\partial X)$. This means that $F := \bar{F}|_X: X \rightarrow Y$ is proper.

Let $E \subseteq X \times X$ be controlled in $T(\bar{X})$. Equivalently, if $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ are nets in X that converge in \bar{X} with $\lim x_i \in \partial X$ or $\lim y_i \in \partial X$ and that satisfy $(x_i, y_i) \in E$ for all $i \in I$, then $\lim x_i = \lim y_i$. Any net $(x_i, y_i)_{i \in I}$ in $(F \times F)(E)$ lifts to a net $(\hat{x}_i, \hat{y}_i)_{i \in I}$ in E . Since E is controlled, accumulation points of $(\hat{x}_i, \hat{y}_i)_{i \in I}$ belong to $X \times X$ or are of the form (x, x) with $x \in \partial X$. Hence all accumulation points of the net $(x_i, y_i)_{i \in I}$ in $(F \times F)(E)$ belong to $Y \times Y \supseteq F(X \times X)$ or are of the form (y, y) with $y \in \partial Y$. Thus $(F \times F)(E)$ is controlled in $T(\bar{Y})$. So F is a coarse map.

Let \bar{F}_1 and \bar{F}_2 be close morphisms, that is, $\partial F_1 = \partial F_2$. If F_1 and F_2 are not close with respect to the coarse structure $T(\bar{Y})$ on Y , then $\{(F_1(x), F_2(x)) \mid x \in X\}$ is not controlled. Hence there is a net $(x_i)_{i \in I}$ in X for which $\lim F_1(x_i)$ and $\lim F_2(x_i)$ exist and are different and one limit belongs to ∂Y . Since \bar{X} is compact, we may pass to a subnet of $(x_i)_{i \in I}$ that converges in \bar{X} . Since F_1 and F_2 are proper, one limit must belong to ∂X . Then $\lim F_1(x_i) = \partial F_1(\lim x_i) = \partial F_2(\lim x_i) = \lim F_2(x_i)$, contradiction. Thus F_1 and F_2 are close. \square

Corollary 6.4. *If the coarse structures on X and Y come from proper metrics or from compactifications with second countable boundary, then a map $F: X \rightarrow Y$ is coarse if and only if it is a morphism in \mathcal{S}_p , and two coarse maps are close if and only if they are close as morphisms in \mathcal{S}_p .*

Proof. Propositions 2.7 and 2.8 say that the functors $H: \mathcal{C} \rightarrow \mathcal{S}_p$ and $T: \mathcal{S}_p \rightarrow \mathcal{C}$ satisfy $H \circ T(X, \bar{X}) = (X, \bar{X})$ and $T \circ H(X, d) = (X, d)$ if $X \subseteq \bar{X}$ is a compactification with second countable boundary or if (X, d) denotes X with the coarse structure of a metric d on X . Now use that both H and T are functors that preserve closeness. \square

What are noncommutative coarse maps in the commutative case? If $\varphi: C_0(X) \rightarrow C_0(Y)$ is a nondegenerate completely positive contraction, then so are the maps $\varphi_y: C_0(X) \rightarrow \mathbb{C}$, $h \mapsto \varphi(h)(y)$, for $y \in Y$. So each φ_y is a state on $C_0(X)$. All states are of the form $h \mapsto \int_X h(x) d\mu_y(x)$ for a unique probability measure μ_y on X . Let $\mathcal{P}(X)$ be the set of probability measures on X with the weak topology defined by the pairing with $C_0(X)$. The functions $\varphi(h)$ for $h \in C_0(X)$ are continuous if and only if the map $Y \rightarrow \mathcal{P}(X)$, $y \mapsto \mu_y$, is continuous. And the function $\varphi(h)$ vanishes at ∞ if and only if $\lim_{y \rightarrow \infty} \int_X h(x) d\mu_y(x) = 0$. So nondegenerate completely positive contractions $C_0(X) \rightarrow C_0(Y)$ are equivalent to continuous maps $\mu: Y \rightarrow \mathcal{P}(X)$ that vanish at ∞ .

Lemma 6.5. *A continuous map $\mu: Y \rightarrow \mathcal{P}(X)$ that vanishes at ∞ corresponds to a noncommutative coarse map from $C_0(X) \triangleleft C(\bar{X})$ to $C_0(Y) \triangleleft C(\bar{Y})$ if and only if the following holds: if a net $(y_i)_{i \in I}$ converges in \bar{Y} to some $y_\infty \in \partial Y$, then there is $x_\infty \in \partial X$ such that $\lim_i \int_X h(x) d\mu_{y_i}(x) = h(x_\infty)$ for all $h \in C(\bar{X})$. In brief,*

$$\lim_{y \rightarrow y_\infty} \int_X h(x) d\mu_y(x) = h(x_\infty)$$

for all $h \in C(\bar{X})$. Two continuous maps $\mu_1, \mu_2: Y \rightarrow \mathcal{P}(X)$ that vanish at ∞ are close as noncommutative coarse maps from $C_0(X) \triangleleft C(\bar{X})$ to $C_0(Y) \triangleleft C(\bar{Y})$ if and only if

$$(3) \quad \lim_{y \rightarrow \infty} \int_X h(x) d\mu_{1,y}(x) - \int_X h(x) d\mu_{2,y}(x) = 0 \quad \text{for all } h \in C(\bar{X}).$$

Proof. The condition in the lemma for a noncommutative coarse map is clearly necessary. Conversely, assume this condition for all convergent nets $(y_i)_{i \in I}$. Merging two nets with the same limit $y_\infty \in \partial Y$, we see that they give the same $x_\infty \in \partial X$. So $y_\infty \mapsto x_\infty$ well-defines a map $\partial\mu: \partial Y \rightarrow \partial X$. If $h \in C(\bar{X})$, then we define $\bar{\varphi}(h): \bar{Y} \rightarrow \mathbb{C}$ by $\bar{\varphi}(h)(y) = \int_X h(x) d\mu_y(x)$ for $y \in Y$ and $\bar{\varphi}(h)(y) = h(\partial\mu(y))$ for $y \in \partial Y$. Then $\lim \bar{\varphi}(h)(y_i) = \bar{\varphi}(h)(\lim y_i)$ whenever $(y_i)_{i \in I}$ is a net in Y that converges in \bar{Y} : if the limit lies in Y , this is the continuity of $y \mapsto \mu_y$, and if the limit lies in ∂Y , it is the construction of $\partial\mu$. Since Y is dense in \bar{Y} , this already implies the continuity of $\bar{\varphi}(h)$ on \bar{Y} (see [2, Lemma 3.30]). So $\bar{\varphi}$ is a noncommutative coarse map. Two maps $\mu_1, \mu_2: C_0(X) \rightarrow C_0(Y)$ are close if and only if the strictly continuous extension of $\mu_1 - \mu_2$ maps $C(\bar{X})$ to $C_0(Y)$. This is equivalent to (3). \square

Proposition 6.6. *Let $X \subseteq \bar{X}$ and $Y \subseteq \bar{Y}$ be objects of \mathcal{S}_p and let $\bar{F}: \bar{X} \rightarrow \bar{Y}$ be a morphism in \mathcal{S}_p . There is a noncommutative coarse map φ from $C_0(Y) \triangleleft C(\bar{Y})$ to $C_0(X) \triangleleft C(\bar{X})$ so that $\partial\varphi = \partial F^*$. Two morphisms \bar{F}_1 and \bar{F}_2 are close if and only if the corresponding noncommutative coarse maps are close.*

Proof. By assumption, there is a neighbourhood U of the diagonal in $X \times X$ that is controlled with respect to the coarse structure $T(\bar{X})$ associated to \bar{X} . The open subsets $V \subseteq X$ with $V \times V \subseteq U$ cover X . Since X is locally compact and σ -compact,

it is paracompact. So there is a partition of unity $\sum_{i \in \mathbb{N}} \psi_i = 1$ with $(\text{supp } \psi_i)^2 \subseteq U$. Choose $x_i \in \text{supp } \psi_i$ and let $y_i := F(x_i)$ for $i \in \mathbb{N}$. Define

$$(\varphi h)(x) := \sum_{i \in \mathbb{N}} h(y_i) \psi_i(x)$$

for $h \in C_0(Y)$, $x \in X$. We claim that φ is a noncommutative coarse map.

The map φ is a completely positive contraction because $\psi_i \geq 0$ for all $i \in \mathbb{N}$ and $\sum \psi_i = 1$. The value $\varphi(h)(x)$ is a convex combination of $h(F(x_i))$ for those $i \in \mathbb{N}$ with $\psi_i(x) \neq 0$. So $(x, x_i) \in (\text{supp } \psi_i)^2 \subseteq U$. This allows to prove both that $\varphi(C_0(Y)) \subseteq C_0(X)$ and that $\lim \varphi(u_i) = 1$ if (u_i) is an approximate unit for $C_0(Y)$. So φ is a nondegenerate completely positive contraction $C_0(Y) \rightarrow C_0(X)$. If $h \in C(\bar{Y})$, then $|h(F(x)) - h(F(x_i))| \rightarrow 0$ as $x \rightarrow \partial X$ in U . Thus the function $x \mapsto h \circ F(x) - \varphi(h)(x)$ on X vanishes at ∞ . Since the function $h \circ \bar{F}$ on \bar{X} is continuous on ∂X , where \bar{F} is continuous, it follows that $\varphi(h)$ is continuous on ∂X . Since continuity on X is clear, we get $\varphi(C(\bar{Y})) \subseteq C(\bar{X})$. Furthermore, $\partial F^* = \partial \varphi$ as desired. By definition, \bar{F}_1 and \bar{F}_2 are close if and only if $\partial F_1 = \partial F_2$, if and only if $\partial \varphi_1 = \partial \varphi_2$, if and only if φ_1 and φ_2 are close. \square

It is unclear, in general, whether any noncommutative coarse map from $C_0(Y) \triangleleft C(\bar{Y})$ to $C_0(X) \triangleleft C(\bar{X})$ comes from a morphism $\bar{F}: \bar{X} \rightarrow \bar{Y}$ in \mathcal{S}_p . Theorem 8.1 will show this if ∂Y is second countable. Even more, any unital $*$ -homomorphism $C(\partial Y) \rightarrow C(\partial X)$ lifts to a morphism in \mathcal{S}_p . The authors tried without success to prove this for Higson compactifications of metric coarse structures. We mention partial results that we obtained in this direction. The problem whether every noncommutative coarse map $\varphi: C_0(Y) \rightarrow C_0(X)$ lifts to an ordinary coarse map is invariant under coarse equivalence by Proposition 6.6. Any (proper) coarse space is coarsely equivalent to a discrete one. This reduces the problem to the case where X and Y are countable sets equipped with proper metrics. If a noncommutative coarse map does not lift to an ordinary coarse map, then this is witnessed by evaluation at a sequence of points $(x_n)_{n \in \mathbb{N}}$ in X that goes to ∞ . Any subsequence of $(x_n)_{n \in \mathbb{N}}$ still witnesses the non-existence of a lifting. We may pass to a subsequence to arrange that $d(x_n, x_m) > |2^n - 2^m|$ for all $n, m \in \mathbb{N}$. Replacing X by the subset $\{x_n \mid n \in \mathbb{N}\}$, we get another counterexample where $X = \mathbb{N}$ with the discrete coarse structure. This is the unique coarse structure on \mathbb{N} where the Higson compactification is the Stone-Ćech compactification, that is, $C_0(X) \triangleleft C(\bar{X})$ is $C_0(\mathbb{N}) \triangleleft \ell^\infty(\mathbb{N})$.

Composing φ with evaluation at $x \in X$ gives a state on $C_0(Y)$, that is, a probability measure on Y . We may arrange these probability measures to have finite supports that go to ∞ for $x \rightarrow \infty$ without changing $\partial \varphi$. Hence we may assume this without loss of generality. Passing to another subsequence, we may then arrange that the finite supports of these probability measures for different points in $X = \mathbb{N}$ are disjoint. Then we may replace Y by the disjoint union of these supports. So if a non-liftable noncommutative coarse map exists, then it exists in the case where $X = \mathbb{N}$ with the discrete coarse structure, Y is a box space $Y = \bigsqcup Y_n$, and

$$(\varphi h)(n) = \sum_{y \in Y_n} c(n, y) h(n, y)$$

for all $h \in C_0(Y)$, where $c(n, y)$ for $y \in Y_n$ are the point masses of a probability measure on Y_n for each $n \in \mathbb{N}$. The challenge is to show that if such a map φ induces a $*$ -homomorphism on $C(\bar{Y})/C_0(Y) \rightarrow C(\bar{X})/C_0(X)$, then there are points $y_n \in Y_n$ so that $\lim_{n \rightarrow \infty} (\varphi h)(n) - h(n, y_n) = 0$.

7. COARSE MAPS FROM THE CORONA ALGEBRA

In this section, we consider noncommutative coarse spaces with nuclear and separable boundary and σ -compact interior. Among such noncommutative coarse spaces, any $*$ -homomorphism between the boundaries lifts to a noncommutative coarse map, which is automatically unique up to closeness. This follows easily from the following theorem, which compares a given noncommutative coarse space to the cone over its boundary in Example 2.10.

Theorem 7.1. *Let $A \triangleleft \bar{A}$ be a noncommutative coarse space. Assume that $\partial A = B$ is separable and nuclear and that A is σ -unital. There is a noncommutative coarse equivalence from $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ to $A \triangleleft \bar{A}$ that induces the identity map on the corona algebras.*

Proof. We first construct a noncommutative coarse map from $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ to $A \triangleleft \bar{A}$ that induces the identity map on the corona algebras, following ideas of [1]. For this, we need two ingredients. The first is a unital, completely positive section $\sigma: \partial A \rightarrow \bar{A}$ for the quotient map $\pi: \bar{A} \twoheadrightarrow \partial A$, which exists by the Choi–Effros Lifting Theorem because ∂A is separable and nuclear (see [5]). The second ingredient is a sequential approximate unit of A that is quasi-central with respect to $\sigma(\partial A)$. This is an increasing sequence $0 = u_0 \leq u_1 \leq u_2 \leq \dots$ in A , such that $\lim \|u_n a - a\| = 0$ for all $a \in A$ and $\lim \|u_n \sigma(b) - \sigma(b) u_n\| = 0$ for all $b \in B$. This exists because A is σ -unital and B is separable. The quickest way to deduce this from [1, Theorem 1] is by replacing A by the separable C^* -subalgebra A' that is generated by some sequential approximate unit (which exists by the σ -unitality assumption) and $\sigma(B)$.

Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of B such that $\bigcup F_n$ is dense in B ; this exists because B is separable. We also assume that $1 \in F_0$. We are going to specify a subsequence $u'_i := u_{n(i)}$ for some increasing function $\mathbb{N} \rightarrow \mathbb{N}$, $i \mapsto n(i)$, with $n(0) = 0$. Let $e_i := (u'_{i+1} - u'_i)^{1/2}$ for $i \in \mathbb{N}$; this is a sequence of positive elements in A with $\sum_{i=0}^{n-1} e_i^2 = u'_n$. Hence $\sum_{i=0}^{\infty} e_i^2 = 1$ with strict convergence. By a lemma in [1], for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|[e_i, \sigma(b)]\| < \varepsilon$ holds for all $b \in B$ and $i \in \mathbb{N}$ with $\|[u'_i, \sigma(b)]\| + \|[u'_{i+1}, \sigma(b)]\| < \delta$. Now choose δ for $\varepsilon = 2^{-i}$ and then choose $n(i) \geq n(i-1)$ so that $\|[u'_j, \sigma(b)]\| < \delta/2$ for all $j \geq n(i)$ and all $b \in F_i$. For this subsequence $(n(i))_{i \in \mathbb{N}}$, we get

$$\|[e_i, \sigma(b)]\| < 2^{-i} \quad \text{for all } b \in F_i, i \in \mathbb{N}.$$

We define

$$\varphi: C_0(\mathbb{N}, B) \rightarrow A, \quad \varphi(f) := \sum_{i=0}^{\infty} e_i \sigma(f(i)) e_i.$$

We claim that this sum is norm convergent for $f \in C_0(\mathbb{N}, B)$. It suffices to check this for $f \geq 0$. Given $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $\|f(i)\| \leq \varepsilon$ for $i \geq n$. Hence

$$0 \leq \sum_{i=n}^N e_i \sigma(f(i)) e_i \leq \sum_{i=n}^N e_i \|f(i)\| e_i \leq \varepsilon \sum_{i=n}^N e_i^2 = \varepsilon (u'_{N+1} - u'_n) \leq \varepsilon$$

if $N \geq n$. This verifies the Cauchy criterion for convergence. Since σ is completely positive and $e_i \geq 0$, each summand is a completely positive map. Thus so is φ .

Let $1_{[0,n]} \in C_0(\mathbb{N}, B)$ be the characteristic function of $[0, n]$; this is an approximate unit in $C_0(\mathbb{N}, B)$. It is mapped to the sequence $\sum_{i=0}^n e_i^2 = u'_{n+1}$ because σ is unital. Since (u'_{n+1}) is an approximate unit in A , the map φ is nondegenerate. Thus φ extends uniquely to a strictly continuous map from $\mathcal{M}(C_0(\mathbb{N}, B)) \cong \ell^\infty(\mathbb{N}, B)$ to $\mathcal{M}(A)$. This map still has the form $f \mapsto \sum_{i=0}^{\infty} e_i \sigma(f(i)) e_i$ for $f \in \ell^\infty(\mathbb{N}, B)$, now

with strict convergence of the infinite sum. If $b \in F_n$, then $\| [e_i, \sigma(b)] \| < 2^{-i}$ for $i \geq n$. For the constant function with value b , this implies

$$\sigma(b) - \sum_{i=0}^{\infty} e_i \sigma(b) e_i = \sum_{i=0}^{\infty} (e_i^2 \sigma(b) - e_i \sigma(b) e_i) = \sum_{i=0}^{\infty} e_i [e_i, \sigma(b)],$$

which is norm-convergent in A . So $\sigma(b) - \varphi(\text{const}(b)) \in A$ for all $b \in F_n$ for all n , and hence for all $b \in B$; here $\text{const } b$ denotes the constant function $\mathbb{N} \rightarrow B$ with value b . Since φ maps $C_0(\mathbb{N}, B)$ and $\text{const}(b)$ for $b \in B$ into \bar{A} , it maps $C(\mathbb{N}^+, B)$ to \bar{A} . The induced map on the corona algebras is the identity map $B \rightarrow \partial A$.

Next we construct a noncommutative coarse ψ map from $A \triangleleft \bar{A}$ to $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ that induces the identity map $\partial A \rightarrow B$. This is inverse to the map constructed above up to closeness, so both maps form a coarse equivalence. The construction of ψ uses the map φ above, and some extra data, namely, an approximation of the identity map on B by maps of the form $\beta_k \circ \alpha_k$, $k \in \mathbb{N}$, with unital completely positive maps $\alpha_k: B \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ and $\beta_k: \mathbb{M}_{n(k)}(\mathbb{C}) \rightarrow B$ for some matrix sizes $n(k)$ for $k \in \mathbb{N}$; this exists because B is nuclear and separable. Our Ansatz is to define

$$\psi: A \rightarrow C_0(\mathbb{N}, B), \quad \psi(a)(k) := \beta_k \gamma_k(a) \quad \text{for } k \in \mathbb{N}, a \in A,$$

where the maps $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ are completely positive contractions with the following extra properties:

- (1) $\lim_{k \rightarrow \infty} \gamma_k(u'_n) = 0$ for each $n \in \mathbb{N}$;
- (2) $\lim_{n \rightarrow \infty} \gamma_k(u'_n) = 1$ for each $k \in \mathbb{N}$;
- (3) let $\mathcal{M}(\gamma_k): \mathcal{M}(A) \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ be the unique strictly continuous extension of γ_k (see below for its existence); then

$$\lim_{k \rightarrow \infty} |\mathcal{M}(\gamma_k)(\varphi(\text{const } b)) - \alpha_k(b)| = 0$$

for all $b \in B$, where $\varphi: C(\mathbb{N}^+, B) \rightarrow \bar{A} \subseteq \mathcal{M}(A)$ is the map built above.

First we show that ψ is a noncommutative coarse map that induces the identity map on the corona algebra if the maps $(\gamma_k)_{k \in \mathbb{N}}$ have the properties listed above. Since the maps γ_k and β_k are completely positive contractions, so are the maps $\beta_k \gamma_k$. Hence ψ is a well-defined completely positive contraction $A \rightarrow \ell^\infty(\mathbb{N}, B)$. It maps the approximate unit (u'_n) into $C_0(\mathbb{N}, A)$ by property (1). Then

$$\begin{aligned} \psi(a(u'_n)^{1/2})^* \psi(a(u'_n)^{1/2}) &\leq \psi((u'_n)^{1/2} a^* a (u'_n)^{1/2}) \\ &\leq \|a\| \psi((u'_n)^{1/2} (u'_n)^{1/2}) \in C_0(\mathbb{N}, B) \end{aligned}$$

for all $a \in A$ by [12, Lemma 5.3]. Since $C_0(\mathbb{N}, B)$ is an ideal in $\ell^\infty(\mathbb{N}, B)$, this implies $\psi(a(u'_n)^{1/2}) \in C_0(\mathbb{N}, B)$ for $a \in A$, and then $\psi(a) \in C_0(\mathbb{N}, B)$ because $(u'_n)^{1/2}$ is an approximate unit. Thus $\psi(A) \subseteq C_0(\mathbb{N}, B)$.

Since ψ is completely positive, the sequence $\psi(u'_n)$ in $C_0(\mathbb{N}, B)$ is increasing. Property (2) of the maps γ_k says that $\psi(u'_n)(k) \rightarrow 1$ for each $k \in \mathbb{N}$ because $\beta_k(1) = 1$. Thus $\psi(u'_n)$ is an approximate unit in $C_0(\mathbb{N}, B)$, so ψ is nondegenerate. Thus ψ has a unique strictly continuous extension $\mathcal{M}(\psi): \mathcal{M}(A) \rightarrow \mathcal{M}(C_0(\mathbb{N}, B)) \cong \ell^\infty(\mathbb{N}, B)$. Evaluation at $k \in \mathbb{N}$ gives $\mathcal{M}(\psi)(a)(k) = \beta_k \circ \mathcal{M}(\gamma_k)(a)$ for the unique strictly continuous extension $\mathcal{M}(\gamma_k)$ of γ_k .

We have $\bar{A} = A + \varphi(\text{const } B)$ because $\bar{A}/A = \partial A = B$ and $B \ni b \mapsto \varphi(\text{const } b) \in \bar{A}$ is a section for this extension. Since $\psi(A) \subseteq C_0(\mathbb{N}, B)$, we have $\mathcal{M}(\psi)(\bar{A}) \subseteq C(\mathbb{N}^+, B)$ if and only if $\mathcal{M}(\psi)(\varphi(\text{const } b)) \in C(\mathbb{N}^+, B)$ for all $b \in B$. Property (3) implies

$$\lim_{k \rightarrow \infty} \mathcal{M}(\psi)(\varphi(\text{const}(b))) = \lim_{k \rightarrow \infty} \beta_k \alpha_k(b) = b$$

because $\|\beta_k\| \leq 1$ for all $k \in \mathbb{N}$. Thus $\psi(\bar{A}) \subseteq C(\mathbb{N}^+, B)$ and ψ induces the identity map $\partial A \rightarrow B$ on the corona algebras. Hence ψ and φ give a coarse equivalence between $A \triangleleft \bar{A}$ and $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$.

It remains to find maps $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ that satisfy (1)–(3). First we fix $k \in \mathbb{N}$. A completely positive linear map $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ is equivalent to a positive linear functional $\tilde{\gamma}_k: \mathbb{M}_{n(k)}(A) \rightarrow \mathbb{C}$, see [15, Chapter 6]. Moreover, $\|\gamma_k\| = \|\tilde{\gamma}_k\|$. The same construction for B turns the given completely positive unital maps α_k into states $\tilde{\alpha}_k: \mathbb{M}_{n(k)}(B) \rightarrow \mathbb{C}$. Any positive linear functional on a C^* -algebra A extends to the multiplier algebra $\mathcal{M}(A)$ by extending its GNS-representation, and this extension is strictly continuous. Then the corresponding completely positive linear map $A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ is strictly continuous as well. Thus any completely positive map $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ is strictly continuous and extends to $\mathcal{M}(A)$.

The unital, completely positive map $b \mapsto \varphi(\text{const } b)$ from B to \bar{A} is faithful because it induces an isomorphism $B \rightarrow \partial A$. Therefore, a self-adjoint element $b \in \mathbb{M}_{n(k)}(B)$ is positive if and only if $\varphi^{(n(k))}(\text{const } b) \geq 0$. Equivalently, $l(\varphi^{(n(k))}(\text{const } b)) \geq 0$ for every state l on $\mathbb{M}_{n(k)}(A)$, where we also write l for the unique strictly continuous extension of l to $\mathbb{M}_{n(k)}(\mathcal{M}(A))$. Therefore, any state on $\mathbb{M}_{n(k)}(B)$ is contained in the weak*-closed convex hull of the set of states of the form $\varphi^*(l)$ with

$$\varphi^*(l)(b) := l(\varphi^{(n(k))}(\text{const } b))$$

for states l on $\mathbb{M}_{n(k)}(A)$ (see [7, Lemma 3.4.1]). Since this set of states is already convex, any state on $\mathbb{M}_{n(k)}(B)$ is a weak*-limit of states of the form $\varphi^*(l)$ for states l on $\mathbb{M}_{n(k)}(A)$. In particular, we may approximate the state $\tilde{\beta}_k: \mathbb{M}_{n(k)}(B) \rightarrow \mathbb{C}$ pointwise by states of the form $\varphi^*(\tilde{\gamma}_k)$ for a state $\tilde{\gamma}_k: \mathbb{M}_{n(k)}(A) \rightarrow \mathbb{C}$.

Thus we may approximate $\beta_k: B \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ pointwise by maps of the form $\mathcal{M}(\gamma_k) \circ \varphi \circ \text{const}$ with completely positive contractions $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$. Choose the increasing sequence of finite subsets $F_i \subseteq B$ as above. There is a completely positive contraction $\gamma_k: A \rightarrow \mathbb{M}_{n(k)}(\mathbb{C})$ with $|\mathcal{M}(\gamma_k) \circ \varphi(\text{const } b) - \beta_k(b)| < 2^{-k}$ for $b \in F_k$. Choosing such γ_k for each $k \in \mathbb{N}$, we get maps γ_k that verify (3) for all $b \in \bigcup F_i$ and hence for all $b \in B$. Since the map $\tilde{\gamma}_k: \mathbb{M}_{n(k)}(A) \rightarrow \mathbb{C}$ corresponding to γ_k is a state, $\lim_{i \rightarrow \infty} \tilde{\gamma}_k(u'_i \cdot 1_{\mathbb{M}_{n(k)}(\mathbb{C})}) = 1$. This is equivalent to $\lim_{i \rightarrow \infty} \gamma_k(u'_i) = 1$. Thus (2) is also built into our construction. To also verify (1), we refine the construction above slightly. The map $\varphi \circ \text{const}: B \rightarrow \bar{A}$ remains faithful when we project to $\mathcal{M}(A)/A$. Hence for each $i \geq 0$, $\varepsilon > 0$, states l of $\mathbb{M}_{n(k)}(A)$ with $l(u'_i) < \varepsilon$ still detect whether self-adjoint elements of $\mathbb{M}_{n(k)}(B)$ are positive. So we may choose γ_k above so that, say, $\gamma_k(u'_i) < 2^{-k}$. Then also $\gamma_k(u'_i) < 2^{-k}$ for $i \leq k$ because $u'_i \leq u'_k$. Thus we have also arranged for (1) to hold. \square

In particular, if ∂A is separable and commutative, then Theorem 7.1 gives a coarse equivalence between $A \triangleleft \bar{A}$ and a commutative coarse space. Our proof needs separability of ∂A in a crucial way. It seems unlikely that similar results hold for non-separable commutative boundaries.

Theorem 7.2. *Let $A_1 \triangleleft \bar{A}_1$ and $A_2 \triangleleft \bar{A}_2$ be noncommutative coarse spaces. Assume that the corona algebra ∂A_1 is nuclear and separable and that A_1 and A_2 are σ -unital. Then any $*$ -homomorphism $f: \partial A_1 \rightarrow \partial A_2$ lifts to a noncommutative coarse map from $A_1 \triangleleft \bar{A}_1$ to $A_2 \triangleleft \bar{A}_2$. All such liftings are close.*

Proof. Theorem 7.1 gives a coarse equivalence between $A_1 \triangleleft \bar{A}_1$ and $C_0(\mathbb{N}, \partial A_1) \triangleleft C(\mathbb{N}^+, \partial A_1)$. Since ∂A_1 is separable and nuclear, the Choi–Effros Lifting Theorem lifts the $*$ -homomorphism $f: \partial A_1 \rightarrow \partial A_2$ to a completely positive, unital map $\sigma_f: \partial A_1 \rightarrow \bar{A}_2$. Let $(u_n)_{n \in \mathbb{N}}$ be a sequential approximate unit for A_2 . Let $A'_2 \subseteq A_2$ be the C^* -subalgebra generated by elements of the form $u_n \cdot \sigma_f(x)$ and $\sigma_f(x) \cdot u_n$ for $n \in \mathbb{N}$, $x \in \partial A_1$. Since ∂A_1 is separable, so is A'_2 . The approximate unit (u_n) lies

in A'_2 and is an approximate unit there, so the map $A'_2 \hookrightarrow A_2$ is nondegenerate and induces an inclusion $\mathcal{M}(A'_2) \hookrightarrow \mathcal{M}(A_2)$. The elements $\sigma_f(x) \in \overline{A_2}$ for $x \in \partial A_1$ are also multipliers of A'_2 . So $\overline{A'_2} = A'_2 + \sigma_f(B) \subseteq \overline{A_2}$ gives another noncommutative coarse space $A'_2 \triangleleft \overline{A'_2}$ contained in $A_2 \triangleleft \overline{A_2}$. Since $\overline{A'_2}/A'_2$ as a quotient of ∂A_1 is separable and nuclear, this new noncommutative coarse space $A'_2 \triangleleft \overline{A'_2}$ is also coarsely equivalent to $C_0(\mathbb{N}, \partial A_1) \triangleleft C(\mathbb{N}^+, \partial A_1)$. Composing the coarse equivalences from $A_1 \triangleleft \overline{A_1}$ to $C_0(\mathbb{N}, \partial A_1) \triangleleft C(\mathbb{N}^+, \partial A_1)$ and on to $A'_2 \triangleleft \overline{A'_2}$ with the inclusion $A'_2 \hookrightarrow A_2$ gives the desired lifting of f . Two noncommutative coarse maps are close if and only if they induce the same map on the boundaries. \square

Remark 7.3. The first part of the proof of Theorem 7.1 still gives a noncommutative coarse map from $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ to $A \triangleleft \overline{A}$ that induces the identity map on the corona algebras, assuming only that there is a completely positive unital section $\partial A \rightarrow \overline{A}$ and a sequential approximate unit for A that is quasi-central in \overline{A} . These sufficient assumptions are also necessary. First, since the boundary quotient map $C(\mathbb{N}^+, B) \twoheadrightarrow B$ has an obvious completely positive, unital section for any B , a noncommutative coarse map from $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ to $A \triangleleft \overline{A}$ that induces the identity map on the boundary can only exist if $\overline{A} \twoheadrightarrow \partial A$ has a completely positive, unital section. Secondly, $(1_{[0, n]})_{n \in \mathbb{N}}$ is a quasi-central approximate unit for $C_0(\mathbb{N}, B) \triangleleft C(\mathbb{N}^+, B)$ because B is unital. A noncommutative coarse map to $A \triangleleft \overline{A}$ maps it to an approximate unit for A because it is nondegenerate. A computation using the Stinespring Dilation and that the induced map on the corona algebras is an isomorphism shows that the image of this approximate unit is quasi-central with respect to \overline{A} .

If \overline{A} is commutative and A σ -unital, then there certainly exists a quasi-central approximate unit. We do not know, however, whether there is a completely positive section $\partial A \rightarrow \overline{A}$: the Choi–Effros Lifting Theorem only applies if ∂A is separable.

8. LIFTING MAPS BETWEEN METRISABLE BOUNDARIES

The following theorem is a version of Theorem 7.2 for ordinary coarse spaces. We have not seen this in the literature. Recall the category of compactified spaces \mathcal{S}_p introduced in Section 6.

Theorem 8.1. *Let $X \subseteq \overline{X}$ and $Y \subseteq \overline{Y}$ be objects of \mathcal{S}_p . Let $\varphi: \partial X \rightarrow \partial Y$ be a continuous map. If the boundary ∂Y is metrisable, then there is a morphism $F: X \rightarrow Y$ in \mathcal{S}_p with boundary values φ , and any two such morphisms are close. The map F is coarse with respect to the coarse structures defined by the compactifications.*

Proof. It suffices to produce morphisms in \mathcal{S}_p . These are coarse maps by Lemma 6.3. We need to find a map $F: X \rightarrow Y$ such that the map $\overline{F}: \overline{X} \rightarrow \overline{Y}$ given by F on X and φ on ∂X is continuous on ∂X . By definition, two morphisms in \mathcal{S}_p are close if and only if they have the same boundary values. Thus any two such morphisms are close. The construction of F needs some preparations. At first, we assume \overline{X} and \overline{Y} to be second countable. We will remove these extra assumptions later.

By assumption, the topologies on \overline{X} and \overline{Y} are metrisable, that is, they may be defined by metrics $d_{\overline{X}}$ and $d_{\overline{Y}}$ on \overline{X} and \overline{Y} , respectively. Since X and Y are σ -compact, there are increasing sequences of compact subsets $(K_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ with $K_0 = \emptyset$, $X := \bigcup_n K_n$, $L_0 = \emptyset$, $Y := \bigcup_n L_n$. We fix these. The balls

$$B^{\partial Y}(y, 2^{-n}) := \{y' \in \partial Y : d_{\overline{Y}}(y', y) < 2^{-n}\}$$

for $y \in \partial Y$ and fixed n form an open cover of ∂Y . Since ∂Y is compact, there are finitely many points $y_{n,1}, y_{n,2}, \dots, y_{n,l}$ such that $\partial Y = \bigcup_{i=1}^l B^{\partial Y}(y_{n,i}, 2^{-n})$. Since $Y \subseteq \overline{Y}$ is dense, $Y \setminus L_n$ is dense in $\overline{Y} \setminus L_n$. Since $B^{\partial Y}(y_{n,i}, 2^{-n})$ for $i \in \{1, \dots, l\}$ is

an open neighbourhood of $y_{n,i} \in \bar{Y} \setminus L_n$, there is $y'_{i,n} \in (Y \setminus L_n) \cap B^{\partial Y}(y_{n,i}, 2^{-n})$. Now we can define the map $F: X \rightarrow Y$. Let $x \in X$. Since $K_0 = \emptyset$ and $X := \bigcup_n K_n$, there is a unique $n \in \mathbb{N}$ with $x \in K_{n+1} \setminus K_n$. We first choose some point $\delta(x)$ in ∂X that is closest to x with respect to the metric $d_{\bar{X}}$. Since $\partial Y = \bigcup_{i=1}^l B^{\partial Y}(y_{n,i}, 2^{-n})$, there is $i \in \{1, \dots, l\}$ with $d_{\bar{Y}}(\varphi(\delta(x)), y_{n,i}) < 2^{-n}$. We pick such an i and let $F(x) := y'_{n,i} \in Y$. This construction is illustrated in Figure 1.

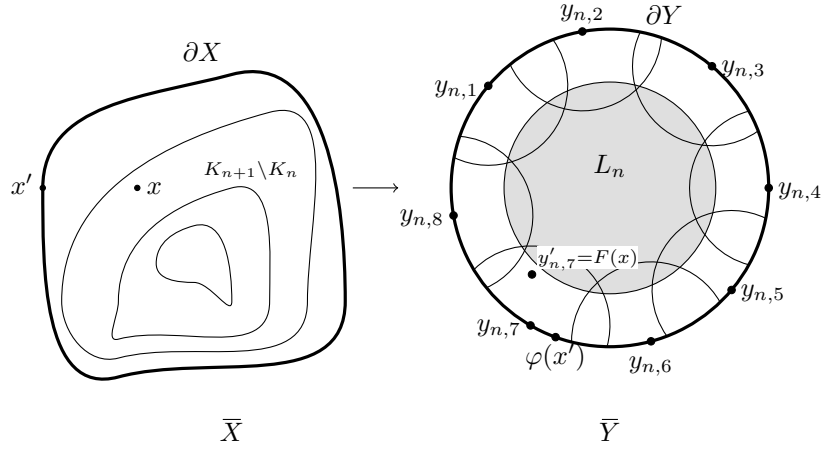


FIGURE 1. Extension of a continuous map φ from the boundary to a map F on the interior.

We claim that the map $\bar{F}: \bar{X} \rightarrow \bar{Y}$ defined by $\bar{F}(x) = F(x)$ if $x \in X$ and $\bar{F}(x) = \varphi(x)$ if $x \in \partial X$ is continuous on ∂X . Let $x_\infty \in \partial X$. We prove continuity at x_∞ . Since \bar{X} is metrisable, it suffices to check $\lim \bar{F}(x_n) = \bar{F}(x_\infty) = \varphi(x_\infty)$ for any sequence $(x_n)_{n \in \mathbb{N}}$ in \bar{X} converging to x_∞ . Since φ is continuous, it suffices to consider sequences in X . Since $\lim d_{\bar{X}}(x_n, x_\infty) = 0$, the points $\delta(x_n) \in \partial X$ closest to x_n chosen in the construction of F satisfy $\lim d_{\bar{X}}(\delta(x_n), x_\infty) = 0$. Hence $\lim d_{\bar{Y}}(\varphi(\delta(x_n)), \varphi(x_\infty)) = 0$. If $x_n \in K_{m+1} \setminus K_m$, then $d_{\bar{Y}}(F(x_n), \varphi(\delta(x_n))) \leq 2^{-m}$. Since $(x_n)_{n \in \mathbb{N}}$ converges to a boundary point, $m \rightarrow \infty$ as $n \rightarrow \infty$. Thus both $d_{\bar{Y}}(F(x_n), \varphi(\delta(x_n)))$ and $d_{\bar{Y}}(\varphi(\delta(x_n)), \varphi(x_\infty))$ converge to 0 as $n \rightarrow \infty$. The triangle inequality shows that $\lim F(x_n) = \varphi(x_\infty)$ as desired.

Now we generalise the result to the case where X and Y are σ -compact and ∂Y is second countable. We have relegated the more technical parts of the proof to the appendix. The image of $C(\partial Y)$ in $C(\partial X)$ is a separable C^* -subalgebra, which corresponds to a second countable quotient of ∂X as in Lemma A.1. Lemma A.1.(1) gives a compactification $X' \subseteq \bar{X}'$ and a continuous quotient map $\varrho: \bar{X} \rightarrow \bar{X}'$ with $\varrho(X) \subseteq X'$ and $\varrho(\partial X) \subseteq \partial X'$ such that \bar{X}' is second countable and such that $\varrho^*C(\partial X') \subseteq C(\partial X)$ is the image $\varphi^*(C(\partial Y))$ of $C(\partial Y)$. Thus $\varphi: \partial X \rightarrow \partial Y$ factors through a continuous map $\varphi': \partial X' \rightarrow \partial Y$. If we can extend φ' to a map $\bar{F}': \bar{X}' \rightarrow \bar{Y}$ that is continuous on $\partial X'$, then $\bar{F} := \bar{F}' \circ \varrho: \bar{X} \rightarrow \bar{Y}$ is the desired extension of φ that is continuous on ∂X . Thus it suffices to prove the theorem for $X' \subseteq \bar{X}'$ instead of $X \subseteq \bar{X}$. Thus it is no loss of generality to assume \bar{X} to be second countable.

Similarly, since Y is σ -compact and ∂Y is second countable, Lemma A.1.(1) gives a compactification $Y' \subseteq \bar{Y}'$ and a continuous quotient map $\varrho: \bar{Y} \rightarrow \bar{Y}'$ with $\varrho(Y) \subseteq Y'$ such that \bar{Y}' is second countable and ϱ restricts to a homeomorphism

$\partial Y \xrightarrow{\sim} \partial Y'$. The proof above applies to $Y' \subseteq \bar{Y}'$, which is second countable, so we get a map $\bar{F}': \bar{X} \rightarrow \bar{Y}'$ with $\bar{F}'(X) \subseteq Y'$ that extends $\varrho \circ \varphi$ on ∂X and that is continuous on ∂X . Since $Y \rightarrow Y'$ is surjective, we may lift \bar{F}' to a map $\bar{F}: \bar{X} \rightarrow \bar{Y}$. This extends φ on ∂X because $\varrho|_{\partial Y}$ is bijective. We claim that it is continuous on ∂X . It suffices to check that $\lim F(x_\alpha) = \varphi(x_\infty)$ for any net (x_α) in X with $\lim x_\alpha = x_\infty \in \partial X$. Since \bar{F}' is continuous, $\lim \varrho \circ F(x_\alpha) = \varrho \circ \varphi(x_\infty)$. This implies $\lim F(x_\alpha) = \varphi(x_\infty)$ by Lemma A.1.(2). \square

APPENDIX A. COMPACTIFICATIONS WITH NON-METRISABLE INTERIORS

The lemmas below help to extend Proposition 2.7 and Theorem 8.1 from second countable compactifications to compactifications with σ -compact interior and second countable boundary.

Lemma A.1. *Let X be a σ -compact, locally compact space with a compactification \bar{X} . Let ∂Y be a second countable quotient of the boundary ∂X .*

- (1) *There are a compactification $X' \subseteq \bar{X}'$ and a continuous map $\varrho: \bar{X} \rightarrow \bar{X}'$ with $\varrho(X) \subseteq X'$ and $\varrho(\partial X) \subseteq \partial X'$, such that \bar{X}' is second countable and the induced map $\partial \varrho: \partial X \rightarrow \partial X'$ is the given map $\partial X \rightarrow \partial Y$.*
- (2) *Assume $\partial X = \partial Y$ and let $X' \subseteq \bar{X}'$ and ϱ be as in (1). Let $x_\infty \in \partial X$ and let $(x_\alpha)_{\alpha \in I}$ be a net in X . If $\lim \varrho(x_\alpha) = \varrho(x_\infty)$, then $\lim x_\alpha = x_\infty$.*
- (3) *If $\partial X = \partial Y$ is second countable, then any $x_\infty \in \partial X$ is the limit of a convergent sequence (x_n) in X .*

Proof. We prove (1). Since ∂Y is second countable, $C(\partial Y)$ contains a dense sequence $(f_n)_{n \in \mathbb{N}}$. The restriction map $C(\bar{X}) \rightarrow C(\partial X)$ is surjective by the Tietze Extension Theorem. Therefore, there are functions $g_n \in C(\bar{X})$ for $n \in \mathbb{N}$ with $g_n|_{\partial X} = f_n$. Since X is σ -compact, there is a countable approximate unit $(u_n^X)_{n \in \mathbb{N}}$ in $C_0(X)$. Let $\bar{A} \subseteq C(\bar{X})$ be the C^* -subalgebra generated by $\{1, g_n, u_n^X \mid n \in \mathbb{N}\}$. Let $\partial A \subseteq C(\partial X)$ be the image of \bar{A} under the quotient map $C(\bar{X}) \rightarrow C(\partial X)$, and let $A = \ker(\partial A \rightarrow \partial A)$. By construction, \bar{A} and ∂A are unital, separable, and commutative. So

$$A \cong C(X'), \quad \bar{A} = C(\bar{X}'), \quad \partial A = C(\partial X')$$

for a second countable, compact space \bar{X}' , an open subspace $X' \subseteq \bar{X}'$, and $\partial X' := \bar{X}' \setminus X'$. By construction, we have a morphism of extensions

$$\begin{array}{ccccc} C_0(X) & \hookrightarrow & C(\bar{X}) & \twoheadrightarrow & C(\partial X) \\ \uparrow & & \uparrow & & \uparrow \\ C_0(X') & \hookrightarrow & C(\bar{X}') & \twoheadrightarrow & C(\partial X'). \end{array}$$

The inclusion map $C_0(X') \hookrightarrow C_0(X)$ is nondegenerate because its image contains an approximate unit for $C_0(X)$. Since the extension in the top row is essential, so is that in the bottom row. That is, $X' \subseteq \bar{X}'$ is dense. So \bar{X}' is a compactification of X' with boundary $\partial X'$. The inclusion map $C(\bar{X}') \hookrightarrow C(\bar{X})$ corresponds to a quotient map $\varrho: \bar{X} \rightarrow \bar{X}'$, which maps X to X' and ∂X to $\partial X'$. By construction, $C(\partial Y) = C(\partial X') \subseteq C(\partial X)$. So we may identify $\partial X' = \partial Y$.

We prove (2). Any quotient as in (1) comes from a compactification $A \hookrightarrow \bar{A} \twoheadrightarrow \partial A$ as in the proof of (1), so we may assume this. Since A is separable, X' is metrisable, so we may define its topology by a metric d' . Then $\varrho^*(d')$ is a continuous quasi-metric on X . Since $\lim \varrho(x_\alpha) = \varrho(x_\infty) \in \partial X'$, we have $\lim f(\varrho(x_\alpha)) = 0$ for all $f \in A$. This implies $\lim f(x_\alpha) = 0$ for all $f \in C_0(X)$ because $\varrho^*: A \rightarrow C_0(X)$ is nondegenerate. This rules out an accumulation point of (x_α) in X . Let $y \in \partial X \setminus \{x_\infty\}$. Then $\varrho(y) \neq$

$\varrho(x_\infty)$ because $\varrho|_{\partial X}$ is a homeomorphism. By Urysohn's Lemma, there is $f \in C(\overline{X}')$ with $f(\varrho(y)) = 1$ and $f(\varrho(x_\infty)) = 0$. Then $\lim f(\varrho(x_\alpha)) = f(\varrho(x_\infty)) \neq f(\varrho(y))$. So (x_α) cannot accumulate at y . Thus x_∞ is the only possible accumulation point of (x_α) . Since \overline{X} is compact, this implies that $\lim x_\alpha = x_\infty$.

Finally, we prove (3). We construct $X' \subseteq \overline{X}'$ as in (1). There is a sequence (x'_n) in X' with $\lim x'_n = \varrho(x_\infty)$ because \overline{X}' is second countable. Since $X \rightarrow X'$ is surjective, we may lift it to a sequence (x_n) in X with $\lim \varrho(x_n) = \varrho(x_\infty)$. Now (2) gives $\lim x_n = x_\infty$. \square

Lemma A.2. *Let X be a σ -compact, locally compact, Hausdorff space and let $X \subseteq \overline{X}$ be a compactification with metrisable boundary ∂X . The corresponding topological coarse structure on X has a controlled neighbourhood of the diagonal.*

Proof. Construct a second countable compactification $X' \subseteq \overline{X}'$ and a continuous quotient map $\varrho: \overline{X} \rightarrow \overline{X}'$ as in Lemma A.1.(1). We may equip \overline{X}' with a metric d' that defines its topology. Let $\varrho^*(d')$ be the resulting continuous pseudo-metric on \overline{X} . Since X is σ -compact, there is an increasing family of relatively compact, open subsets $U_n \subseteq X$ with $\bigcup U_n = X$. For $n \in \mathbb{N}$, let

$$E_n := \{(x, y) \in U_n \times U_n : d(x, y) < 2^{-n}\}.$$

This is open in $U_n \times U_n$ for each n because d is continuous on $\overline{X} \times \overline{X}$, and it contains the diagonal of U_n . Thus the union $E = \bigcup_n E_n$ is an open neighbourhood of the diagonal in X . We claim that E is controlled. So let (x_α, y_α) be a net in E that converges in $\overline{X} \times \overline{X}$, and assume, say, that $\lim x_\alpha \in \partial X$. We must prove $\lim x_\alpha = \lim y_\alpha$. Choose $n(\alpha) \in \mathbb{N}$ minimal with $x_\alpha \in U_{n(\alpha)}$. Then $\lim n(\alpha) = \infty$ because $\lim x_\alpha \in \partial X$. And $d'(\varrho(x_\alpha), \varrho(y_\alpha)) = d(x_\alpha, y_\alpha) < 2^{-n(\alpha)}$ because $(x_\alpha, y_\alpha) \in E$. Thus $\lim \varrho(x_\alpha) = \lim \varrho(y_\alpha)$ because d' defines the topology on \overline{X}' . Now Lemma A.1.(2) gives $\lim x_\alpha = \lim y_\alpha$. \square

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